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[B—179]

PREFACE.

I HAVE to acknowledge most gratefully the assistance rendered me in the preparation of this book by Mr. T. H. Gascoigne, son of the Rev. T. Gascoigne, of Spondon House School, Derby. For the solutions of a few of the Problems I am indebted to Mr. Gaskin's *Trigonometrical Examples*, and to Mr. Hymers' *Trigonometry*. I shall be glad to receive corrections of errors that may be discovered in my work.

CAMBRIDGE, *October* 1876.

ELEMENTARY TRIGONOMETRY.

KEY.

EXAMPLES—I. (pp. 1, 2).

(1) 4 feet 6 inches = 54 inches; ∴ number is 54.

(2) 15 feet 2 inches = 182 inches; ∴ number is $182 \div 7$, or, 26.

(3) Unit of square measurement is $(192 \div 12)$ square inches, or, 16 square inches; ∴ unit of linear measurement is $\sqrt{16}$ inches, or, 4 inches.

(4) Unit of square measurement is $(1000 \div 40)$ square inches, or, 25 square inches; ∴ unit of linear measurement is $\sqrt{25}$ inches, or, 5 inches.

(5) Unit of cubic measurement is $(216 \div 8)$ cubic inches, or, 27 cubic inches; ∴ unit of linear measurement is $\sqrt[3]{27}$ inches, or, 3 inches.

(6) Unit of cubic measurement is $(2000 \div 16)$ cubic inches, or, 125 cubic inches; ∴ unit of linear measurement is $\sqrt[3]{125}$ inches, or, 5 inches.

(7) Measure of 1 yard is $\frac{1}{a}$;

∴ measure of 1 foot is $\frac{1}{3a}$;

∴ measure of b feet is $\frac{b}{3a}$.

(8) Length of line is (5×6) inches, or, 30 inches; $\therefore$ second unit is $(30 \div 4)$ inches, or, $7\frac{1}{2}$ inches.

(9) Length of line is 1 yard, or, 36 inches;
$\therefore$ second unit is $(36 \div 36)$ inches, or, 1 inch;
and third unit is $(36 \div 12)$ inches, or, 3 inches.

(10) The ratio is $3\frac{1}{4} : 3\frac{1}{2} \times 36$, or, $3\frac{1}{4} : 126$, or, $13 : 126 \times 4$, or, $13 : 504$.

(11) The measure of 1 yard is $\frac{c}{m}$;

$\therefore$ the measure of 1 foot is $\frac{c}{3m}$;

$\therefore$ the measure of n feet is $\frac{nc}{3m}$.

EXAMPLES—II. (p. 3).

(1) Length of other side $= \sqrt{(51)^2 - (24)^2}$ yards $= \sqrt{2601 - 576}$ yards $= \sqrt{2025}$ yards $= 45$ yards.

(2) Length of hypothenuse $= \sqrt{8^2 + 6^2}$ feet $= \sqrt{64 + 36}$ feet $= \sqrt{100}$ feet $= 10$ feet.

(3) Diagonal $= \sqrt{\{(225)^2 + (120)^2\}}$ yards $= \sqrt{65025}$ yards $= 255$ yards.

(4) Diagonal $= \sqrt{\{(300)^2 + (200)^2\}}$ yards $= \sqrt{130000}$ yards $= 360{\cdot}5 \ldots$ yards.

(5) Length $= \frac{2\frac{1}{2} \times 4840}{88}$ yards $= (2{\cdot}5 \times 55)$ yards $= 137{\cdot}5$ yards;

diagonal $= \sqrt{\{(137{\cdot}5)^2 + (88)^2\}}$ yards $= \sqrt{26650{\cdot}25}$ yards $= 163{\cdot}25$ yards, nearly.

(6) Let $x+y$, x, $x-y$ be the length of the sides in feet.

$$\text{Then } (x+y)^2 = x^2 + (x-y)^2,$$
$$\text{or, } x^2 + 2xy + y^2 = x^2 + x^2 - 2xy + y^2,$$
$$\text{or, } 4xy = x^2, \text{ and } \therefore x = 4y.$$

4

Hence $x + y = 5y$,
and $5y = 20$ feet, and $\therefore y = 4$ feet.
Hence the other sides are 16 feet and 12 feet.

(7) Proceeding as in Example (6), we get

$$x+y=5y\,;\ x=4y\,;\ x-y=3y.$$

Hence the sides are as $3y:4y:5y$, that is, as $3:4:5$.

(8) Let $AB=36$ feet, and $DE=27$ feet;

CA, CE being the two positions of the ladder.

Then since ACE is a right angle, $\angle ACB$, ECD, are together equal to a right angle.

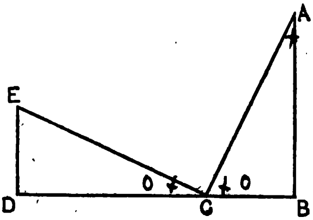


FIG. 1.

But $\angle^s ACB$, CAB are together equal to a right angle,

and $\therefore\ \angle ECD=\angle CAB$.

Hence in $\triangle$s ABC, CDE.

right $\angle ABC=$right $\angle CDE$, and $\angle CAB=\angle ECD$, and $AC=CE$;

$\therefore\ BC=ED$, and $AB=CD$; (EUCLID, I. XXVI.)

$\therefore$ width of street $=(27+36)$ feet $=63$ feet,

and length of ladder $=\sqrt{(27)^2+(36)^2}$ feet $=\sqrt{2025}$ feet $=45$ feet.

(9) Let $x=$length of each of the equal sides in feet.

Then $x^2+x^2=(12)^2$, or, $2x^2=144$, or, $x^2=72$, or, $x=6\sqrt{2}$.

(10) Diagonal$=\sqrt{25+25}$ inches $=\sqrt{50}$ inches $=5\sqrt{2}$ inches.

(11) Each side of square $=\sqrt{390625}$ feet $=625$ feet;

$\therefore$ diagonal$=\sqrt{\{(625)^2+(625)^2\}}$ feet$=\sqrt{\{2\,.\,(625)^2\}}$ feet$=625\sqrt{2}$. feet.

(12) AD bisects BC.

$\therefore\ AB=2.BD$.

Let $x=$measure of length of AD.

$$\text{Then } x^2=(13)^2-\left(\frac{13}{2}\right)^2$$

$$=(13)^2\{1-\tfrac{1}{4}\}=(13)^2\cdot\tfrac{3}{4}.$$

$$\therefore\ x=\frac{13\sqrt{3}}{2}.$$

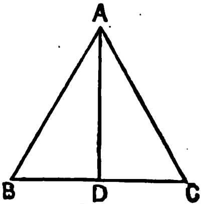


FIG 2.

(13) Taking the diagram in Example 12, let measure of AB be x.

Then $x^2=\frac{x^2}{4}+(15)^2$;

$\therefore\ 3x^2=4\times(15)^2$, or, $x^2=\frac{4\times(15)^2}{3}=\frac{4\times(15)^2\times 3}{9}$;

$\therefore\ x=\frac{2\times 15\sqrt{3}}{3}=10\sqrt{3}$.

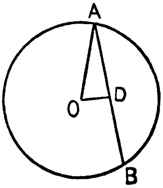


Fig. 3.

(14) OD, a perpendicular from the centre on the chord AB, bisects AB.

Let x = measure of OD in inches.

Then $x^2=(OA)^2-(AD)^2$

$=(37)^2-(35)^2=144$;

$\therefore$ distance $=\sqrt{144}$ inches $=12$ inches.

(15) Taking the diagram of Example (14).

Let measure of AD in inches be x.

Then $x^2=(181)^2-(180)^2=361$;

$\therefore\ x=19$, and $\therefore\ AB=(2\times 19)$ inches $=38$ inches.

(16) Taking the diagram of Example (14).

Let measure of AO in feet be x.

Then $x^2=(308)^2+(75)^2=100489$;

$\therefore\ x=317$, and $\therefore$ diameter $=(2\times 317)$ feet $=634$ feet.

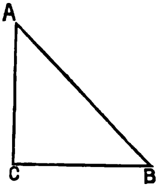


Fig. 4.

(17) $(AC)^2+(BC)^2=(AB)^2$.

$\therefore\ 2(AC)^2=(AB)^2$;

$\therefore\ \frac{(AC)^2}{(AB)^2}=\frac{1}{2}$;

$\therefore\ \frac{AC}{AB}=\frac{1}{\sqrt{2}}$.

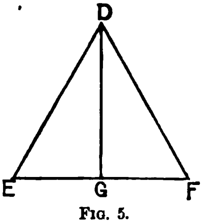


Fig. 5.

(18) Let x be the measure of EG.

Then $2x$ is the measure of ED;

and measure of $DG=\sqrt{(4x^2-x^2)}=\sqrt{3}\,.\,x$;

$\therefore\ EG:ED:DG=x:2x:\sqrt{3}\,.\,x$

$=1:2:\sqrt{3}$.

EXAMPLES—III. (p. 9).

(1) Circumference $=\frac{22\times 5}{7}$ feet $=\frac{110}{7}$ feet $=15\frac{5}{7}$ feet.

(2) Radius $=\frac{7\times 542\cdot 5}{44}$ feet $=\frac{3797\cdot 5}{44}$ feet $=86\cdot 306\dot{8}\dot{1}$ feet.

(3) Train goes in a second $\frac{22\times 12}{7}$ feet.

Rate in miles per hour $=\frac{22\times 12\times 60\times 60}{7\times 3\times 1760}=\frac{180}{7}=25\cdot\dot{7}1428\dot{5}$.

(4) Diameter in miles $=\frac{7\times 25000}{22}=7954\frac{6}{11}$.

(5) Circumference in miles $=\frac{22\times 883220}{7}=2775834\frac{2}{7}$.

(6) Radius in miles $=\frac{7\times 6850}{44}=\frac{23975}{22}=1089\frac{17}{22}$.

(7) Circumference in feet $=\frac{22\times 12\frac{1}{2}\times 2}{7}=\frac{22\times 25}{7}$;

$\therefore$ $\frac{1}{12}$ of circumference $=\frac{22\times 25}{12\times 7}$ feet $=6$ feet $6\frac{4}{7}$ inches.

(8) Circumference in feet $=\frac{22\times 21}{7}$;

$\therefore$ $\frac{5}{7}$ of circumference $=\frac{22\times 21\times 5}{7\times 7}$ feet $=47\frac{1}{7}$ feet.

(9) If x be the side of the square,

$(\text{diameter})^2=2x^2$;

$$\therefore\ x=\frac{\text{diameter}}{\sqrt{2}}=\frac{7\times 150}{22\times\sqrt{2}}\text{ feet}=\frac{7\times 150\times\sqrt{2}}{22\times\sqrt{2}\times\sqrt{2}}\text{ feet}=\frac{525\sqrt{2}}{22}\text{ feet.}$$

$$(10)\ x=\frac{\text{diameter}}{\sqrt{2}}=\frac{7\times 200}{22\times\sqrt{2}}\text{ feet}=\frac{7\times 200\times\sqrt{2}}{22\times 2}\text{ feet}=\frac{350\sqrt{2}}{11}\text{ feet.}$$

(11) Point goes in a minute $\frac{22\times 12\times 30}{7}$ feet.

Rate in miles per hour $=\frac{22\times 12\times 30\times 60}{7\times 1760\times 3}=\frac{90}{7}=12\frac{6}{7}$.

(12) End goes in a minute $\frac{22 \times 2 \times 15 \times 21}{7}$ feet.

$$\text{Rate in miles per hour} = \frac{22 \times 2 \times 15 \times 21 \times 60}{7 \times 3 \times 1760} = \frac{45}{2} = 22\tfrac{1}{2}.$$

EXAMPLES—IV. (p. 12).

(1)
$$\begin{array}{r|l} 60 & 5{\cdot} \\ \hline 60 & 16{\cdot}08\dot{3} \\ \hline & {\cdot}2680\dot{5} \end{array}$$
$\therefore$ $24^\circ.\,16'.\,5'' = 24^\circ{\cdot}2680\dot{5}$

(2)
$$\begin{array}{r|l} 60 & 43{\cdot} \\ \hline 60 & 2{\cdot}71\dot{6} \\ \hline & {\cdot}0452\dot{7} \end{array}$$
$\therefore$ $37^\circ.\,2'.\,43'' = 37^\circ{\cdot}0452\dot{7}.$

(3)
$$\begin{array}{r|l} 60 & 14{\cdot} \\ \hline 60 & 0{\cdot}2\dot{3} \\ \hline & {\cdot}003\dot{8} \end{array}$$
$\therefore$ $175^\circ.\,0'.\,14'' = 175{\cdot}003\dot{8}.$

(4)
$$\begin{array}{r|l} 60 & 28{\cdot} \\ \hline 60 & 5{\cdot}4\dot{6} \\ \hline & {\cdot}09\dot{1} \end{array}$$
$\therefore$ $5'.\,28'' = {\cdot}09\dot{1}^\circ.$

(5)
$$\begin{array}{r|l} 60 & 4{\cdot} \\ \hline & {\cdot}0\dot{6} \end{array}$$
$\therefore$ $375^\circ.\,4' = 375^\circ{\cdot}0\dot{6}.$

(6)
$$\begin{array}{r|l} 60 & 4{\cdot} \\ \hline 60 & 12{\cdot}0\dot{6} \\ \hline & {\cdot}20\dot{1} \end{array}$$
$\therefore$ $78^\circ.\,12'.\,4'' = 78^\circ{\cdot}20\dot{1}.$

EXAMPLES—V. (p. 13).

(1) $25^g.\,14'.\,25'' = 25_g{\cdot}1425.$

(2) $38^g.\,\ 4'.\,15'' = 38^g{\cdot}0415.$

(3) $214^g.\,3'.\,\ 7'' = 214^g{\cdot}0307.$

(4) $15'.\,7''{\cdot}45 = {\cdot}150745^g.$

(5) $425^g.\,13'.\,5''{\cdot}54 = 425^g{\cdot}130554.$

(6) $2^g.\,2'.\,2''{\cdot}22 = 2^g{\cdot}020222.$

EXAMPLES—VI. (p. 19).

(1) $27^\circ.\,15'.\,46'' = 27^\circ{\cdot}262\dot{7}$

$$\begin{array}{l} \quad\ 10 \\ \hline |\ 272{\cdot}62\dot{7} \\ \hline \ 30{\cdot}291975\ \ldots \end{array}$$

$\therefore$ $27^\circ.\,15'.\,46'' = 30^g.\,29'.\,19''{\cdot}75\ \ldots$

(2) $157^\circ.4'.9'' = 157^\circ\cdot 0691\dot{6}$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 1570\cdot 691\dot{6} \\ \hline & 174\cdot 5212\dot{9}6\dot{2} \end{array}$$

$\therefore\ 157^\circ.\,4'.\,9'' = 174^{g}.\,52^{\backprime}.\,12^{\backprime\backprime}\cdot\dot{9}6\dot{2}$.

(3) $24'.\,18'' = 0^\circ\cdot 405$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 4\cdot 05 \\ \hline & \cdot 45 \end{array}$$

$\therefore\ 24'.\,18'' = 45^{\backprime}$.

(4) $19^\circ.0'.18'' = 19^\circ\cdot 005$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 190\cdot 05 \\ \hline & 21\cdot 1166\dot{6} \end{array}$$

$\therefore\ 19^\circ.\,0'.\,18'' = 21^{g}.\,11^{\backprime}.\,66^{\backprime\backprime}\cdot\dot{6}$.

(5) $143^\circ.9' = 143^\circ\cdot 15$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 1431\cdot 5 \\ \hline & 159\cdot 0555\dot{5} \end{array}$$

$\therefore\ 143^\circ.\,9' = 159^{g}.\,5^{\backprime}.\,55^{\backprime\backprime}\cdot\dot{5}$.

(6) $28^\circ = 28^\circ$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 280 \\ \hline & 31\cdot 1111\dot{1} \end{array}$$

$\therefore\ 28^\circ = 31^{g}.\,11'.\,11''\cdot\dot{1}$.

(7) $10^\circ.\,25'.\,48'' = 10^\circ\cdot 43$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 104\cdot 3 \\ \hline & 11\cdot 5888\dot{8} \end{array}$$

$\therefore\ 10^\circ.\,25'.\,48'' = 11^{g}.\,58^{\backprime}.\,88^{\backprime\backprime}\cdot\dot{8}$.

$$(8)\ 27^{\circ}.\,38'.\,12''=27^{\circ}\cdot 63\dot{6}$$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 276\cdot 3\dot{6} \\ \hline & 30\cdot 7\dot{0}7\dot{4} \end{array}$$

$$\therefore\ 27^{\circ}.\,38'.\,12''=30^{g}.\,70^{\backprime}.\,74^{\backprime\backprime}\cdot\dot{0}7\dot{4}.$$

$$(9)\ 300^{\circ}.\,15'.\,58''=300^{\circ}\cdot 266\dot{1}$$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 3002\cdot 66\dot{1} \\ \hline & 333\cdot 6290\dot{1}23456789\dot{0} \end{array}$$

$$\therefore\ 300^{\circ}.\,15'.\,58''=333^{g}.\,62^{\backprime}.\,90^{\backprime\backprime}\cdot\dot{1}23456789\dot{0}.$$

$$(10)\ 422^{\circ}.\,7'.\,22''=422^{\circ}\cdot 122\dot{7}$$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 4221\cdot 22\dot{7} \\ \hline & 469\cdot 0253\dot{0}8641975\dot{3} \end{array}$$

$$\therefore\ 422^{\circ}.\,7'.\,22''=469^{g}.\,2^{\backprime}.\,53^{\backprime\backprime}\cdot\dot{0}8641975\dot{3}.$$

EXAMPLES—VII. (p. 20).

$$(1)\ 19^{g}.\,45^{\backprime}.\,95^{\backprime\backprime}=19^{g}\cdot 4595$$

$$\begin{array}{lr|l} & & 9 \\ \hline & 10 & 175\cdot 1355 \\ \hline \text{degrees} & & 17\cdot 51355 \\ & & 60 \\ \hline \text{minutes} & & 30\cdot 81300 \\ & & 60 \\ \hline \text{seconds} & & 48\cdot 78000 \end{array}$$

$$\therefore\ 19^{g}.\,45^{\backprime}.\,95^{\backprime\backprime}=17^{\circ}.\,30'.\,48''\cdot 78.$$

$$(2)\ 124^{g}.\,5^{\backprime}.\,8^{\backprime\backprime}=124^{g}\cdot 0508$$

$$\begin{array}{lr|l} & & 9 \\ \hline & 10 & 1116\cdot 4572 \\ \hline \text{degrees} & & 111\cdot 64572 \\ & & 60 \\ \hline \text{minutes} & & 38\cdot 74320 \\ & & 60 \\ \hline \text{seconds} & & 44\cdot 59200 \end{array}$$

$$\therefore\ 124^{g}.\,5^{\backprime}.\,8^{\backprime\backprime}=111^{\circ}.\,38'.\,44''\cdot 592.$$

(3) $29^g.\,75' = 29^g{\cdot}75$

$$\begin{array}{lr}
 & 9 \\
10 \,| & \overline{\underline{267{\cdot}75}} \\
\text{degrees} & 26{\cdot}775 \\
 & 60 \\ \hline
\text{minutes} & 46{\cdot}500 \\
 & 60 \\ \hline
\text{seconds} & 30{\cdot}000
\end{array}$$

$\therefore\ 29^g.\,75' = 26^\circ.\,46'.\,30''.$

(4) $15^g.\,0'.\,15'' = 15^g{\cdot}0015$

$$\begin{array}{lr}
 & 9 \\
10 \,| & \overline{\underline{135{\cdot}0135}} \\
\text{degrees} & 13{\cdot}50135 \\
 & 60 \\ \hline
\text{minutes} & 30{\cdot}08100 \\
 & 60 \\ \hline
\text{seconds} & 4{\cdot}86000
\end{array}$$

$\therefore\ 15^g.\,0'.\,15'' = 13^\circ.\,30'.\,4''{\cdot}86.$

(5) $154^g.\,7'.\,24'' = 154^g{\cdot}0724$

$$\begin{array}{lr}
 & 9 \\
10 \,| & \overline{\underline{1386{\cdot}6516}} \\
\text{degrees} & 138{\cdot}66516 \\
 & 60 \\ \hline
\text{minutes} & 39{\cdot}90960 \\
 & 60 \\ \hline
\text{seconds} & 54{\cdot}57600
\end{array}$$

$\therefore\ 154g.\,7'.\,24'' = 138^\circ.\,39'.\,54''{\cdot}576.$

(6) $43^g = 43_g$

$$\begin{array}{lr}
 & 9 \\
10 \,| & \overline{\underline{387}} \\
\text{degrees} & 38{\cdot}7 \\
 & 60 \\ \hline
\text{minutes} & 42{\cdot}0
\end{array}$$

$\therefore\ 43^g = 38^\circ.\,42'.$

(7) $38^g.\,71'.\,20''\cdot3 = 38^g\cdot71203$

$$\begin{array}{lr} & 9 \\ \hline 10\,| & 348\cdot40827 \\ \hline \text{degrees} & 34\cdot840827 \\ & 60 \\ \hline \text{minutes} & 50\cdot449620 \\ & 60 \\ \hline \text{seconds} & 26\cdot977200 \end{array}$$

$\therefore\ 38^g.\,71'.\,20''\cdot3 = 34^\circ.\,50'.\,26''\cdot9772.$

(8) $50^g.\,76'.\,94''\cdot3 = 50^g\cdot76943$

$$\begin{array}{lr} & 9 \\ \hline 10\,| & 456\cdot92487 \\ \hline \text{degrees} & 45\cdot692487 \\ & 60 \\ \hline \text{minutes} & 41\cdot549220 \\ & 60 \\ \hline \text{seconds} & 32\cdot953200 \end{array}$$

$\therefore\ 50^g.\,76'.\,94''\cdot3 = 45^\circ.\,41'.\,32''\cdot9532.$

(9) $170^g.\,63'.\,27'' = 170^g\cdot6327$

$$\begin{array}{lr} & 9 \\ \hline 10\,| & 1535\cdot6943 \\ \hline \text{degrees} & 153\cdot56943 \\ & 60 \\ \hline \text{minutes} & 34\cdot16580 \\ & 60 \\ \hline \text{seconds} & 9\cdot94800 \end{array}$$

$\therefore\ 170^g.\,63'.\,27'' = 153^\circ.\,34'.\,9''\cdot948.$

(10) $324^g.\,13'.\,88''\cdot7 = 324^g\cdot13887$

$$\begin{array}{lr} & 9 \\ \hline 10\,| & 2917\cdot24983 \\ \hline \text{degrees} & 291\cdot724983 \\ & 60 \\ \hline \text{minutes} & 43\cdot498980 \\ & 60 \\ \hline \text{seconds} & 29\cdot938800 \end{array}$$

$\therefore\ 324^g.\,13'.\,88''\cdot7 = 291^\circ.\,43'.\,29''\cdot9388.$

EXAMPLES—VIII. (p. 21).

(1) Circular measure is $\frac{60\times\pi}{180}=\frac{\pi}{3}$.

(2) Circular measure is $\frac{22\cdot5\times\pi}{180}=\frac{\pi}{8}$.

(3) Circular measure is $\frac{11\cdot25\times\pi}{180}=\frac{\pi}{16}$.

(4) Circular measure is $\frac{270\times\pi}{180}=\frac{3\pi}{2}$.

(5) Circular measure is $\frac{315\times\pi}{180}=\frac{7\pi}{4}$.

(6) Circular measure is $\frac{24\frac{13}{60}\times\pi}{180}=\frac{1453\pi}{60\times180}=\frac{1453\pi}{10800}$.

(7) Circular measure is $\frac{95\frac{1}{3}\times\pi}{180}=\frac{286\times\pi}{180\times3}=\frac{143\pi}{270}$.

(8) Circular measure is $\frac{12\frac{304}{3600}\times\pi}{180}=\frac{43504\times\pi}{180\times3600}=\frac{2719\pi}{40500}$.

(9) Circular measure of each angle is $\frac{60\times\pi}{180}=\frac{\pi}{3}$.

(10) The angles are 90°, 45°, 45°, and of these the circular measures are $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$.

EXAMPLES—IX. (p. 22).

(1) Measure in degrees is $\frac{\pi\times180}{2\times\pi}=90$.

(2) Measure in degrees is $\frac{\pi\times180}{3\times\pi}=60$.

(3) Measure in degrees is $\frac{\pi\times180}{4\times\pi}=45$.

(4) Measure in degrees is $\frac{\pi \times 180}{6 \times \pi} = 30$.

(5) Measure in degrees is $\frac{2\pi \times 180}{3 \times \pi} = 120$.

(6) Measure in degrees is $\frac{1 \times 180}{2 \times \pi} = \frac{90}{\pi}$.

(7) Measure in degrees is $\frac{1 \times 180}{3 \times \pi} = \frac{60}{\pi}$.

(8) Measure in degrees is $\frac{1 \times 180}{4 \times \pi} = \frac{45}{\pi}$.

(9) Measure in degrees is $\frac{1 \times 180}{6 \times \pi} = \frac{30}{\pi}$.

(10) Measure in degrees is $\frac{2 \times 180}{3 \times \pi} = \frac{120}{\pi}$.

EXAMPLES—X. (p. 22).

(1) Circular measure is $\frac{50 \times \pi}{200} = \frac{\pi}{4}$.

(2) Circular measure is $\frac{25 \times \pi}{200} = \frac{\pi}{8}$.

(3) Circular measure is $\frac{6{\cdot}25 \times \pi}{200} = \frac{\pi}{32}$.

(4) Circular measure is $\frac{250 \times \pi}{200} = \frac{5\pi}{4}$.

(5) Circular measure is $\frac{500 \times \pi}{200} = \frac{5\pi}{2}$.

(6) Circular measure is $\frac{13{\cdot}0505 \times \pi}{200} = {\cdot}0652525\pi$.

(7) Circular measure is $\frac{24{\cdot}150215 \times \pi}{200} = {\cdot}120751075\pi$.

(8) Circular measure is $\frac{125{\cdot}0013 \times \pi}{200} = {\cdot}6250065\pi$.

(9) Circular measure is $\frac{\cdot 03\times\pi}{200}=\cdot 00015\pi$.

(10) Circular measure is $\frac{\cdot 0005\times\pi}{200}=\cdot 0000025\pi$.

EXAMPLES—XI. (p. 22).

(1) Measure in grades is $\frac{\pi\times 200}{3\times\pi}=66\cdot\dot{6}$.

(2) Measure in grades is $\frac{\pi\times 200}{5\times\pi}=40$.

(3) Measure in grades is $\frac{\pi\times 200}{6\times\pi}=33\cdot\dot{3}$.

(4) Measure in grades is $\frac{2\pi\times 200}{3\times\pi}=133\cdot\dot{3}$.

(5) Measure in grades is $\frac{3\pi\times 200}{5\times\pi}=120$.

(6) Measure in grades is $\frac{1\times 200}{3\times\pi}=\frac{200}{3\pi}$.

(7) Measure in grades is $\frac{1\times 200}{5\times\pi}=\frac{40}{\pi}$.

(8) Measure in grades is $\frac{1\times 200}{8\times\pi}=\frac{25}{\pi}$.

(9) Measure in grades is $\frac{3\times 200}{5\times\pi}=\frac{120}{\pi}$.

(10) Measure in grades is $\frac{23\times 200}{10\times\pi}=\frac{460}{\pi}$.

EXAMPLES—XII. (p. 23).

(1) Measure $=22\frac{1}{2}\div 5=22\cdot 5\div 5=4\cdot 5$.

(2) Unit $=42\cdot 5^\circ\div 10=4\cdot 25^\circ$.

(3) Angle $=8\times 2^\circ$, or, 16°;

$\therefore$ larger unit $=16^\circ\div 5=3\frac{1}{5}^\circ$.

Then, smaller unit in terms of larger is $2\div 3\frac{1}{5}$, or, $\frac{5}{8}$,

and larger unit in terms of smaller is $3\frac{1}{5}\div 2$, or, $\frac{8}{5}$.

(4) Angle $= 7 \times 3^\circ$, or, 21°;

$\therefore$ larger unit $= 21^\circ \div 6 = 3\frac{1}{2}^\circ$.

Then, smaller unit in terms of larger is $3 \div 3\frac{1}{2}$, or, $\frac{6}{7}$,

and larger unit in terms of smaller is $3\frac{1}{2} \div 3$, or, $\frac{7}{6}$.

(5) Measure $= 42 \div 45 = \frac{14}{15}$.

(6) $13^\circ.\,13'.\,48'' = 47628''$,

$$14^g.\,7' = \frac{227934''}{5};$$

$\therefore$ ratio $= 47628 \times 5 : 227934 = 70 : 67$.

(7) $G : D = 10 : 9$, $\therefore$, $9G = 10D$, $\therefore$, $G = D + \frac{1}{9}D$.

(8) The angles of each triangle are 90°, 60°, 30°, because the line, drawn from any angle of an equilateral triangle to bisect the base, cuts the base at right angles, and bisects the vertical angle.

Expressed in grades the angles are 100^g, $66\frac{2}{3}^g$, $33\frac{1}{3}^g$.

(9) Let $x+y$, x, $x-y$ be the angles.

Then $x+y+x+x-y = 180^\circ$, or, $3x = 180^\circ$, or, $x = 60^\circ$.

(10)

$$\begin{array}{lr} & 39^g\cdot012 \\ & 9 \\ \hline 10\,| & 351\cdot108 \\ \hline \text{degrees} & 35\cdot1108 \\ & 60 \\ \hline \text{minutes} & 6\cdot6480 \\ & 60 \\ \hline \text{seconds} & 38\cdot8800 \end{array}$$

(11) Number of degrees in the angle $= \frac{m}{60}$.

Number of grades ,, $= \frac{m \times 10}{60 \times 9}$.

Number of French seconds ,, $= \frac{m \times 10 \times 100 \times 100}{60 \times 9} = \frac{5000m}{27}$.

(12) $5^\circ.\,33'.\,20'' = 20000''$; and $90^\circ = 324000''$;

$\therefore$ fraction $= \frac{20000}{5} \div 324000 = \frac{4000}{324000} = \frac{1}{81}$.

(13) Let x be the measure of the angle in degrees.

Then $\frac{10x}{9}$ is the measure of the angle in grades,

and $\frac{1}{x}+\frac{9}{10x}=1$, or, $10x=19$, or, $x=1{\cdot}9$;

$\therefore$ unit angle is $1{\cdot}9^\circ$.

(14) Let $x+y$, x, $x-y$ be the angles expressed in degrees.

Then $\frac{10(x+y)}{9}=x+(x-y)$;

or, $10x+10y=18x-9y$, and $\therefore\ x=\frac{19y}{8}$;

$\therefore$ the angles are $\frac{27y}{8}$, $\frac{19y}{8}$, $\frac{11y}{8}$,

and these are in the ratio 27 : 19 : 11.

(15) $\frac{180^\circ}{\sqrt{3}}=\frac{10\times180^{g.}}{9\times\sqrt{3}}=\frac{200^{g}}{\sqrt{3}}=\frac{200\sqrt{3}^{g.}}{3}=115^{g}.47'$ nearly.

(16) Let $x+y$, x, $x-y$ be the angles expressed in degrees.
Then $x+y+x+x-y=180^\circ$, or, $3x=180^\circ$, or, $x=60^\circ$.
Also $\frac{10}{9}(60-y):60+y=2:9$;
or, $600-10y=120+2y$, and $\therefore\ y=40^\circ$.
Hence the angles are 100°, 60°, 20°.

(17) Circumference : diameter $=360:2\times57{\cdot}29577$
$=180:57{\cdot}29577$
$=3{\cdot}14159\ldots:1$.

(18) The sum of the two angles is 90°, because the third angle is 90°.
Hence, dividing 90° into two parts proportional to 2 and 3, we have 36° and 54° for the angles.
$\therefore$ angles expressed in degrees are 90°, 54°, 36°.
" " grades are 100^{g}, 60^{g}, 40^{g}.
" " circular measure are $\frac{\pi}{2}$, $\frac{3\pi}{10}$, $\frac{\pi}{5}$.

(19) Angle : $360^\circ=13:27$;
$\therefore$ angle $=\frac{360\times13}{27}$ degrees $=\frac{40\times13}{3}$ degrees $=173\frac{1}{3}^\circ$.

(20) Angle : $400^g = 17 : 54$;

$$\therefore \text{angle} = \frac{400 \times 17}{54} \text{ grades} = 125\cdot\dot{9}2\dot{5} \text{ grades.}$$

(21) Angle subtended by an arc 18 inches long = unit of circular

$$\text{measure} = \frac{200}{\pi} \text{ grades;}$$

$$\therefore \text{angle subtended by an arc 24 inches long} = \frac{24 \times 200}{18 \times \pi} \text{ gr.} = \frac{800}{3\pi} \text{ gr.}$$

(22) 1st angle contains $\frac{2 \times 200}{\pi}$ grades, or, $\frac{400}{\pi}$ grades.

2d angle contains $\frac{10 \times 20}{9}$ grades, or, $\frac{200}{9}$ grades.

3d angle contains $\left(200 - \frac{400}{\pi} - \frac{200}{9}\right)$gr., or, $\frac{1600\pi - 3600}{9\pi}$ gr.

(23) Angle required $= \frac{7}{2}$ of $15°.39'.7'' = 54°.46'.54''\cdot5$.

(24) Circular measure $= \frac{11\cdot3 \times \pi}{200} = \frac{113 \times 355}{2000 \times 113} = \frac{71}{400} = \cdot1775$.

(25) Measure in degrees $= \frac{180 \times \pi^2}{\pi \times 9} = 20\pi$.

(26) Larger circumference = 400 times smaller circumference.
Then, since $\frac{1}{360}$th part of smaller circumference subtends an angle of 1° at the centre, it follows that $\frac{1}{400}$ of $\frac{1}{360}$th part of the larger circumference will subtend the same angle.

$$\therefore \text{part required} = \frac{1}{400 \times 360} = \frac{1}{144000}.$$

(27) 4 right angles $= 360° = 400^g = 2\pi^c$;

$$\therefore \text{the measure of } 1° \text{ will be } \frac{1}{360},$$

$$\text{the measure of } 1^g \text{ will be } \frac{1}{400},$$

$$\text{the measure of } 1^c \text{ will be } \frac{1}{2\pi}.$$

(28) Length of whole circumference of earth $= 7980\pi$ miles;

$\therefore$ length of 1 degree of meridian $= \dfrac{7980\pi}{360}$ miles $= \dfrac{133\pi}{6}$ miles.

(29) (1) $\dfrac{3}{2} \times 45^\circ = 67\tfrac{1}{2}^\circ$; $4 \times 45^\circ = 180^\circ$; $\pi \times 45^\circ = 45\pi^\circ$;

$\left(4n + \dfrac{1}{3}\right) \times 45^\circ = (n\,.\,180 + 15)^\circ$.

(2) $\dfrac{3}{2} \times \dfrac{\pi}{4} = \dfrac{3\pi}{8}$; $4 \times \dfrac{\pi}{4} = \pi$; $\pi \times \dfrac{\pi}{4} = \left(\dfrac{\pi}{2}\right)^2$;

$\left(4n + \dfrac{1}{3}\right) \times \dfrac{\pi}{4} = n\pi + \dfrac{\pi}{12}$.

(30) Number of degrees in the unit angle $= \dfrac{3 \times 180}{\pi}$;

$\therefore$ measure of an angle of $45^\circ = 45 \div \dfrac{3 \times 180}{\pi} = \dfrac{45 \times \pi}{3 \times 180} = \dfrac{\pi}{12}$.

(31) (1) Sum of angles $= (12 - 4)$ right angles $= 8$ right angles.
(EUCLID, I. XXXII., COR. 1.)

$\therefore$ each angle $= \dfrac{8 \times 90}{6}$ degrees $= 120^\circ$.

(2) Sum of angles $= (10 - 4)$ right angles $= 6$ right angles;

$\therefore$ each angle $= \dfrac{6 \times 90}{5}$ degrees $= 108^\circ$.

(32) (1) Each angle $= \dfrac{6 \times 100}{5}$ grades $= 120^g$.

(2) Each angle $= \dfrac{12 \times 100}{8}$ grades $= 150^g$.

(33) (1) Circular measure of each angle $= \dfrac{\pi}{3}$.

(2) Circular measure of each angle $= \dfrac{8 \times \pi}{6 \times 2} = \dfrac{2\pi}{3}$.

(34) Sum of all the angles $= (2n - 4)$ right angles;

$\therefore$ circular measure of each angle $= \dfrac{2n - 4}{n} \cdot \dfrac{\pi}{2} = \pi - \dfrac{2\pi}{n}$.

(35) Arc subtending an angle of $180^\circ = 18\pi$ feet.

$\therefore$ arc subtending an angle of $10^\circ = \dfrac{18\pi}{18}$ feet $= \pi$ feet.

2

B

(36) Let $2n$ and n be the number of sides in the polygons, respectively.

Each angle in first polygon contains $\frac{4n-4}{2n}$ right angles.

Each angle in second polygon contains $\frac{2n-4}{n}$ right angles.

$$\therefore \frac{4n-4}{2n} : \frac{2n-4}{n} = 3:2;$$

$$\therefore 4n-4=6n-12, \text{ or, } 2n=8, \text{ or, } n=4.$$

Hence the number of sides will be 8 and 4 respectively.

EXAMPLES—XIII. (p. 36).

(1) $\sin BAD=\frac{BD}{AB}$; $\cos BAD=\frac{AD}{AB}$; $\tan BAD=\frac{BD}{AD}$;

$\sin ABD=\frac{AD}{AB}$; $\cot ABD=\frac{BD}{AD}$; $\operatorname{cosec} ABD=\frac{AB}{AD}$;

$\sin BCD=\frac{BD}{BC}$; $\sin CBD=\frac{CD}{BC}$; $\tan BCD=\frac{DB}{DC}$.

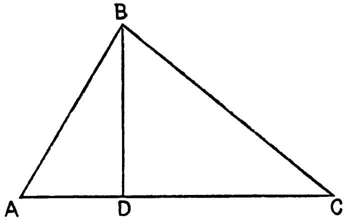


FIG. 6.

(2) $\frac{a}{b}=\sin A, \therefore a=b.\sin A,$

$\frac{a}{b}=\cos C, \therefore a=b.\cos C,$

$\frac{a}{c}=\tan A, \therefore a=c.\tan A,$

$\frac{a}{c}=\cot C, \therefore a=c.\cot C;$

and similarly for the rest of the Examples.

EXAMPLES—XIV. (p. 49).

(1) $\cos\alpha \,.\, \sin\gamma \,.\, \cos\delta = \cos 0^\circ \,.\, \sin 45^\circ \,.\, \cos 60^\circ = 1 \times \frac{1}{\sqrt{2}} \,.\, \frac{1}{2} = \frac{1}{2\sqrt{2}}.$

(2) $\sin\theta . \cos\frac{\pi}{4} . \operatorname{cosec}\delta = \sin 90^\circ \,.\, \cos 45^\circ \,.\, \operatorname{cosec} 60^\circ$

$$= 1 \times \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{3}} = \sqrt{\frac{2}{3}}.$$

(3) $\sin\frac{\pi}{2} + \cos\frac{\pi}{6} - \sec\alpha = \sin 90^\circ + \cos 30^\circ - \sec 0^\circ = 1 + \frac{\sqrt{3}}{2} - 1 = \frac{\sqrt{3}}{2}.$

(4) $\sin\frac{\pi}{3} \,.\, \operatorname{cosec}\frac{\pi}{2} . \sec\delta = \sin 60^\circ . \operatorname{cosec} 90^\circ . \sec 60^\circ = \frac{\sqrt{3}}{2} \times 1 \times 2 = \sqrt{3}.$

(5) $(\sin\theta - \cos\theta + \operatorname{cosec}\beta)\left(\cos\theta + \sec\frac{\pi}{4} + \cot\delta\right)$

$$= (\sin 90^\circ - \cos 90^\circ + \operatorname{cosec} 30^\circ).(\cos 90^\circ + \sec 45^\circ + \cot 60^\circ)$$

$$= (1 - 0 + 2).\left(0 + \sqrt{2} + \frac{1}{\sqrt{3}}\right) = 3 \times \left(\sqrt{2} + \frac{1}{\sqrt{3}}\right) = 3\sqrt{2} + \sqrt{3}.$$

(6) $(\sin\delta - \sin\gamma)(\cos\beta + \cos\gamma) = (\sin 60^\circ - \sin 45^\circ)(\cos 30^\circ + \cos 45^\circ)$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4},$$

$$\sin^2\beta = \sin^2 30^\circ = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

(7) $\cot^2\frac{\pi}{4} - \cot^2\frac{\pi}{6} = \cot^2 45^\circ - \cot^2 30^\circ = 1 - 3 = -2.$

$$\frac{\sin^2\frac{\pi}{6} - \sin^2\frac{\pi}{4}}{\sin^2\frac{\pi}{4} \cdot \sin^2\frac{\pi}{6}} = \frac{\frac{1}{4} - \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2}} = \frac{-2}{1} = -2.$$

(8) $\left(\sin\frac{\pi}{6}+\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{3}-\cos\frac{\pi}{3}\right)=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$

$$=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}=\cos\frac{\pi}{3}.$$

(9) $\cos\frac{\pi}{3}\cdot\cos\frac{\pi}{6}=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}.$

$$\frac{1}{2}\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+\frac{1}{2}\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{1}{2}\cos\frac{\pi}{2}+\frac{1}{2}\cos\frac{\pi}{6}$$

$$=\frac{1}{2}\times 0+\frac{1}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}.$$

(10) $\tan^2\frac{\pi}{3}-\tan^2\frac{\pi}{6}=3-\frac{1}{3}=\frac{8}{3}.$

$$\frac{\sin^2\frac{\pi}{3}-\sin^2\frac{\pi}{6}}{\cos^2\frac{\pi}{3}\cdot\cos^2\frac{\pi}{6}}=\frac{\frac{3}{4}-\frac{1}{4}}{\frac{1}{4}\cdot\frac{3}{4}}=\frac{8}{3}.$$

EXAMPLES—XV. (p. 52).

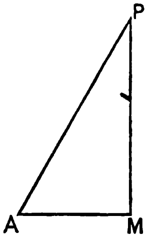


FIG. 7.

(1.) Let PM be the tower; A the place of observation.

Then $AM=200$ feet, and $\angle PAM=60°$.

Now $PM=AM.\tan PAM$

$=(200\times\sqrt{3})$ feet $=346{\cdot}4101\ldots$ feet.

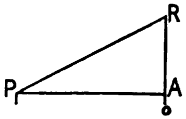


FIG. 8.

(2) Let RO be the tower; P the point of observation.

Then $AP=140$ feet, and $\angle RPA=30°$.

Now $RA=PA.\tan 30°$.

$=\frac{140}{\sqrt{3}}$ feet $=\frac{140\sqrt{3}}{3}$ feet $=80{\cdot}829037\ldots$ feet.

$\therefore\ RO=80{\cdot}829037\ldots$ feet $+\ 5$ feet $=85{\cdot}829037\ldots$ feet.

(3) Taking the diagram in Art. 87,

$$AB : BQ = \sqrt{3} : 1;$$

$$\therefore \tan SQR = \sqrt{3}, \text{ and } \therefore \angle SQR = 60°.$$

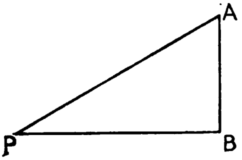


FIG. 9.

(4) Let AB be the steeple; P the point of observation.

Then $PB = 300$ feet, and $\angle APB = 30°$.

Then $AB = PB \cdot \tan APB$

$$= 300 \cdot \frac{1}{\sqrt{3}} \text{ feet} = 100\sqrt{3} \text{ . feet}$$

$$= 173{\cdot}205 \ldots \text{ feet.}$$

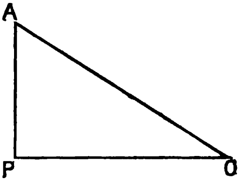


FIG. 10.

(5) Let AP be the rock; O the position of the ship.

Then $AP = 245$ feet; and $\angle AOP = 30°$.

Now $PO = AP \cdot \cot AOP$

$$= 245 \cdot \sqrt{3} \text{ ft.} = 424{\cdot}352 \ldots \text{ ft.}$$

(6) Let AB be the hill; C and D the positions of the milestones.

Then $DC = 1$ mile; $\angle ACB = 45°$;

$$\angle ADB = 30°.$$

Hence $\angle CAB = 45°$, and $AB = BC$.

Let x = height of hill in miles.

Then $AB = BD \cdot \tan ADB$

$$= (BC + CD) \cdot \tan 30°;$$

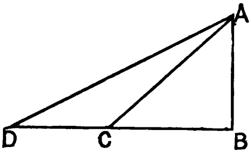


FIG. 11.

$$\therefore x = (x+1) \cdot \frac{1}{\sqrt{3}};$$

$$\therefore \sqrt{3} \cdot x = x + 1, \text{ or, } x = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2};$$

$$\therefore x = \frac{2{\cdot}732 \ldots}{2} \text{ miles} = 1{\cdot}366 \ldots \text{ miles.}$$

(7) Let AO be the flag-staff; PO the tower; M the point of observation.

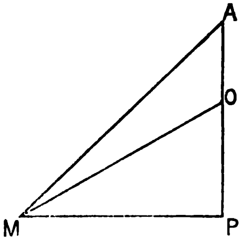


FIG. 12.

Then $PM=100$ feet; $\angle AMP=45°$; $\angle OMP=30°$.

$$\begin{aligned}\text{Then } AO&=AP-OP\\&=PM-PM.\tan OMP\\&=\left(100-100\cdot\frac{1}{\sqrt{3}}\right)\text{ feet.}\\&=\frac{300-100\sqrt{3}}{3}\text{ feet}\\&=\frac{300-173\cdot205\ldots}{3}\text{ feet}=42\cdot265\ldots\text{ feet.}\end{aligned}$$

(8) Let AP be the tower; MO the column; MD parallel to OP.

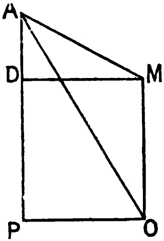


FIG. 13.

Then $\angle AMD=30°$, and $\angle AOP=60°$.

$$\text{Then } MD=OP=AP.\cot AOP=\frac{108}{\sqrt{3}}\text{ feet.}$$

$$\text{And } AD=MD.\tan AMD=\frac{108}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\text{ feet}$$

$$=\frac{108}{3}\text{ feet}=36\text{ feet.}$$

Hence $OM=AP-AD=(108-36)$ feet $=72$ feet.

(9.) Let AP be the tower; M and O the points of observation.

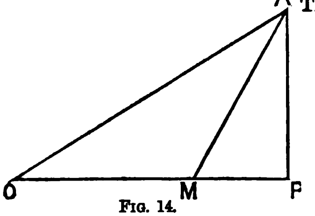


FIG. 14.

Then $\angle AMP=60°$; and $\angle AOP=30°$.

Let $x=$ height of tower in yards.

Now $\angle AMP=\angle AOP+\angle OAM$,

or, $60°=30°+\angle OAM$;

$\therefore\ \angle OAM=30°=\angle AOM$,

and $\therefore\ MA=OM=100$ yards.

$$\text{Then } AP=MA\,.\,\sin AMP=100\cdot\frac{\sqrt{3}}{2}\text{ yards}=50\sqrt{3}\text{ yards.}$$

(10) Taking the diagram in Art. 87.

$$\tan AQB = \frac{AB}{BQ} = \frac{10}{25} = \cdot 4\,;$$

$$\therefore \text{ altitude of the sun is } 25^\circ.$$

(11) The diagram represents a vertical section of the spire and tower.

Let x represent the height of the spire in feet.

Then $AM = x + 35 - 23 = x + 12$,

$BM = 60 + 17\tfrac{1}{2} = 77\cdot 5$,

and $\frac{x+12}{77\cdot 5} = \tan ABM = 1\cdot 5$;

$\therefore\ x + 12 = 116\cdot 25$, or, $x = 104\cdot 25$ feet.

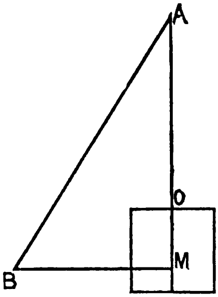


FIG. 15.

(12) Let OB be the height of the kite in yards.

Let AO be the string.

Then $OB = AO\,.\sin OAB$

$= \left(250 \times \frac{1}{2}\right)$yards $= 125$ yards.

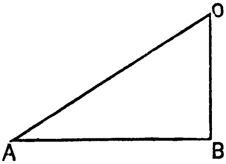


FIG. 16.

(13) Let AC be the rope ; AB the height of the house.

Then $\angle\, ACB = 40^\circ\,.\,30'$.

And $AC = \frac{AB}{\sin ACB} = \frac{60}{\cdot 65}$ feet $= 92\frac{4}{13}$ feet.

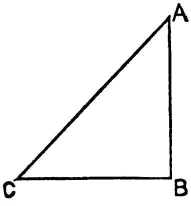


FIG. 17.

(14) Let AC be the tower; BC the breadth of the river.

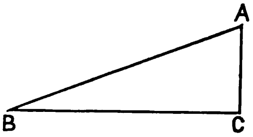


Fig. 18.

Then $\angle ABC = 20°$.

And $BC = \dfrac{AC}{\tan ABC}$

$= \dfrac{120}{\cdot 35}$ feet $= 342\frac{6}{7}$ feet.

(15) Taking the diagram of Art. 87.

Length of shadow $= QB = \dfrac{AB}{\tan AQB} = \dfrac{6}{\cdot 745}$ feet $= 8\cdot 053 \ldots$ feet.

Examples—XVI. (p. 57).

(1) $\cos\theta \cdot \tan\theta = \cos\theta \cdot \dfrac{\sin\theta}{\cos\theta} = \sin\theta.$

(2) $\sin\theta \cdot \cot\theta = \sin\theta \cdot \dfrac{\cos\theta}{\sin\theta} = \cos\theta.$

(3) $\sin a \cdot \sec a = \sin a \cdot \dfrac{1}{\cos a} = \dfrac{\sin a}{\cos a} = \tan a.$

(4) $\cos a \cdot \operatorname{cosec} a = \cos a \cdot \dfrac{1}{\sin a} = \dfrac{\cos a}{\sin a} = \cot a.$

(5) $(1 + \tan^2\theta) \cdot \cos^2\theta = \sec^2\theta \cdot \cos^2\theta = \dfrac{\cos^2\theta}{\cos^2\theta} = 1.$

(6) $(1 + \cot^2\theta) \cdot \sin^2\theta = \operatorname{cosec}^2\theta \cdot \sin^2\theta = \dfrac{\sin^2\theta}{\sin^2\theta} = 1.$

(7) $\dfrac{\tan^2 a}{1 + \tan^2 a} = \dfrac{\tan^2 a}{\sec^2 a} = \dfrac{\sin^2 a}{\cos^2 a} \cdot \cos^2 a = \sin^2 a.$

(8) $\dfrac{\operatorname{cosec}^2 a - 1}{\operatorname{cosec}^2 a} = 1 - \dfrac{1}{\operatorname{cosec}^2 a} = 1 - \sin^2 a = \cos^2 a.$

(9) $\tan x + \cot x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x} = \dfrac{1}{\cos x \cdot \sin x} = \sec x \cdot \operatorname{cosec} x.$

(10) $$\frac{\cos x\,.\,\operatorname{cosec}x\,.\,\tan x}{\sin x\,.\,\sec x\,.\,\cot x}=\frac{\cos x\cdot\dfrac{1}{\sin x}\cdot\dfrac{\sin x}{\cos x}}{\sin x\cdot\dfrac{1}{\cos x}\cdot\dfrac{\cos x}{\sin x}}=\frac{\cos x\,.\,\sin x\,.\,\cos x\,.\,\sin x}{\sin x\,.\,\sin x\,.\,\cos x\,.\,\cos x}=1.$$

(11) $$\cos x+\sin x\,.\,\tan x=\cos x+\frac{\sin^2 x}{\cos x}=\frac{\cos^2 x+\sin^2 x}{\cos x}=\frac{1}{\cos x}=\sec x.$$

(12) $$\frac{\cos\theta}{\tan\theta\,.\,\cot^2\theta}=\frac{\cos\theta}{\cot\theta}=\frac{\cos\theta\,.\,\sin\theta}{\cos\theta}=\sin\theta.$$

(13) $$(\cos^2\theta-1)(\cot^2\theta+1)=(\cos^2\theta-1)\,.\,\operatorname{cosec}^2\theta=-\sin^2\theta\times\frac{1}{\sin^2\theta}=-1.$$

(14) $$\cot^2 a-\cos^2 a=\frac{\cos^2 a}{\sin^2 a}-\cos^2 a=\cos^2 a\left(\frac{1}{\sin^2 a}-1\right)=\cos^2 a\cdot\frac{1-\sin^2 a}{\sin^2 a}$$
$$=\cos^2 a\cdot\frac{\cos^2 a}{\sin^2 a}=\cot^2 a\,.\,\cos^2 a.$$

(15) $$\sec^2 a\,.\,\operatorname{cosec}^2 a=\sec^2 a(1+\cot^2 a)=\sec^2 a+\sec^2 a\cdot\frac{\cos^2 a}{\sin^2 a}$$
$$=\sec^2 a+\operatorname{cosec}^2 a.$$

(16) $$\sin^2\phi+\sin^2\phi\,.\,\tan^2\phi=\sin^2\phi(1+\tan^2\phi)=\sin^2\phi\,.\,\sec^2\phi=\tan^2\phi.$$

(17) $$\cot^2\phi\,.\,\sin^2\phi+\sin^2\phi=\sin^2\phi(\cot^2\phi+1)=\sin^2\phi\,.\,\operatorname{cosec}^2\phi=1.$$

(18) $$\sec^2\phi-1=\tan^2\phi=\frac{\sin^2\phi}{\cos^2\phi}=\sin^2\phi\,.\,\sec^2\phi.$$

(19) $$2\operatorname{versin}\phi-\operatorname{versin}^2\phi=2(1-\cos\phi)-(1-\cos\phi)^2$$
$$=2-2\cos\phi-1+2\cos\phi-\cos^2\phi=1-\cos^2\phi=\sin^2\phi.$$

(20) $$\frac{\sec\theta-1}{\sec\theta}=1-\frac{1}{\sec\theta}=1-\cos\theta=\operatorname{versin}\theta.$$

EXAMPLES—XVII. (p. 60).

(1) Let PAM be an angle whose cosine is c. Draw PM perpendicular to AM. Then if AP be represented by 1, AM will be represented by c, and PM will be represented by $\sqrt{1-c^2}$. Then, denoting $\angle PAM$ by A,

$$\sin A=\frac{PM}{AP}=\frac{\sqrt{1-c^2}}{1}=\sqrt{1-\cos^2 A}$$

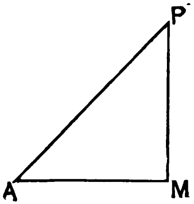


FIG. 19.

$$\tan A=\frac{PM}{AM}=\frac{\sqrt{1-c^2}}{c}=\frac{\sqrt{1-\cos^2 A}}{\cos A}$$

$$\sec A=\frac{AP}{AM}=\frac{1}{c}=\frac{1}{\cos A}$$

$$\operatorname{cosec} A=\frac{AP}{PM}=\frac{1}{\sqrt{1-c^2}}=\frac{1}{\sqrt{1-\cos^2 A}}$$

$$\cot A=\frac{AM}{PM}=\frac{c}{\sqrt{1-c^2}}=\frac{\cos A}{\sqrt{1-\cos^2 A}}$$

(2) Let PAM be an angle whose cosecant is c.

Constructing a diagram as in Example (1), the measures of AP, PM, AM may be taken as c, 1, $\sqrt{c^2-1}$ respectively.

$$\text{Then } \sin A=\frac{PM}{AP}=\frac{1}{c}=\frac{1}{\operatorname{cosec} A}$$

$$\cos A=\frac{AM}{AP}=\frac{\sqrt{c^2-1}}{c}=\frac{\sqrt{\operatorname{cosec}^2 A-1}}{\operatorname{cosec} A}$$

$$\tan A=\frac{PM}{AM}=\frac{1}{\sqrt{c^2-1}}=\frac{1}{\sqrt{\operatorname{cosec}^2 A-1}}$$

$$\sec A=\frac{AP}{AM}=\frac{c}{\sqrt{c^2-1}}=\frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A=1}}$$

$$\cot A=\frac{AM}{MP}=\frac{\sqrt{c^2-1}}{1}=\sqrt{\operatorname{cosec}^2 A-1}.$$

(3) Let PAM be an angle whose secant is s.

Constructing a diagram as in Example (1), the measures of AP, AM, PM, may be taken as s, 1, $\sqrt{s^2-1}$ respectively.

$$\text{Then } \sin A=\frac{PM}{AP}=\frac{\sqrt{s^2-1}}{s}=\frac{\sqrt{\sec^2 A-1}}{\sec A}$$

$$\cos A=\frac{AM}{AP}=\frac{1}{s}=\frac{1}{\sec A}$$

$$\tan A=\frac{PM}{AM}=\frac{\sqrt{s^2-1}}{1}=\sqrt{\sec^2 A-1}$$

$$\operatorname{cosec} A=\frac{AP}{PM}=\frac{s}{\sqrt{s^2-1}}=\frac{\sec A}{\sqrt{\sec^2 A-1}}$$

$$\cot A=\frac{AM}{PM}=\frac{1}{\sqrt{s^2-1}}=\frac{1}{\sqrt{\sec^2 A-1}}.$$

(4) Let *PAM* be an angle whose cotangent is c.

Constructing a diagram as in Example (1), the measures of *AM*, *PM*, *AP* may be taken as c, 1, $\sqrt{1+c^2}$ respectively.

$$\text{Then } \sin A = \frac{PM}{AP} = \frac{1}{\sqrt{1+c^2}} = \frac{1}{\sqrt{1+\cot^2 A}}$$

$$\cos A = \frac{AM}{AP} = \frac{c}{\sqrt{1+c^2}} = \frac{\cot A}{\sqrt{1+\cot^2 A}}$$

$$\tan A = \frac{PM}{AM} = \frac{1}{c} = \frac{1}{\cot A}$$

$$\operatorname{cosec} A = \frac{AP}{PM} = \frac{\sqrt{1+c^2}}{1} = \sqrt{1+\cot^2 A}$$

$$\sec A = \frac{AP}{AM} = \frac{\sqrt{1+c^2}}{c} = \frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

EXAMPLES—XVIII. (p. 61).

(1) Take the diagram as before; then if $\angle PAM$ be denoted by a, the measure of *PM* may be denoted by 2, the measure of *AP* by 3, and therefore the measure of *AM* by $\sqrt{9-4} = \sqrt{5}$.

$$\text{Then } \cos a = \frac{\sqrt{5}}{3} \text{ and } \tan a = \frac{2}{\sqrt{5}}.$$

(2) Let the measure of *AM* be 4, and that of *AP* be 5; then that of *AM* will be $\sqrt{25-16}$, or, 3.

$$\text{Then } \sin a = \frac{3}{5}, \text{ and } \tan a = \frac{3}{4}.$$

(3) Let the measure of *AP* be 4, and that of *PM* be 3; then that of *PM* will be $\sqrt{16-9}$, or, $\sqrt{7}$.

$$\text{Then } \cos\theta = \frac{\sqrt{7}}{4}, \text{ and } \tan\theta = \frac{3}{\sqrt{7}}.$$

(4) Let the measure of *PM* be 1, and that of *AP* be $\sqrt{3}$; then that of *AM* will be $\sqrt{3-1}$, or, $\sqrt{2}$.

$$\text{Then } \cos\theta = \sqrt{\frac{2}{3}}, \text{ and } \tan\theta = \frac{1}{\sqrt{2}}.$$

(5) Let the measure of PM be a^2, and that of AM be b^2; then that of AP will be $\sqrt{a^4+b^4}$.

$$\text{Then } \operatorname{cosec} a = \frac{\sqrt{a^4+b^4}}{a^2}, \text{ and } \sec a = \frac{\sqrt{a^4+b^4}}{b^2}.$$

(6) Let the measure of AM be a, and that of AP be b; then that of PM will be $\sqrt{b^2-a^2}$.

$$\text{Then } \tan a = \frac{\sqrt{b^2-a^2}}{a}, \text{ and } \operatorname{cosec} a = \frac{b}{\sqrt{b^2-a^2}}.$$

(7) Let the measure of PM be a, and that of AP be 1; then that of AM will be $\sqrt{1-a^2}$.

$$\text{Then } \tan\theta = \frac{a}{\sqrt{1-a^2}}, \text{ and } \sec\theta = \frac{1}{\sqrt{1-a^2}}.$$

(8) Let the measure of AM be b, and that of AP be 1; then that of PM will be $\sqrt{1-b^2}$.

$$\text{Then } \tan\theta = \frac{\sqrt{1-b^2}}{b}, \text{ and } \operatorname{cosec}\theta = \frac{1}{\sqrt{1-b^2}}.$$

(9) Let the measure of PM be 6, and that of AP be 10; then that of AM will be $\sqrt{100-36}$, or, 8.

$$\text{Then } \cos\theta = \frac{8}{10} = \frac{4}{5}, \text{ and } \cot\theta = \frac{8}{6} = \frac{4}{3}.$$

(10) Let the measure of AM be 5, and that of AP be 9; then that of PM will be $\sqrt{81-25} = \sqrt{56} = 2\sqrt{14}$.

$$\text{Then } \cot\theta = \frac{5}{2\sqrt{14}}, \text{ and } \operatorname{cosec}\theta = \frac{9}{2\sqrt{14}}.$$

(11) Let the measure of AP be 22, and that of PM be 9; then that of AM will be $\sqrt{484-81} = \sqrt{403}$.

$$\text{Then } \cos\theta = \frac{\sqrt{403}}{22}, \text{ and } \cot\theta = \frac{\sqrt{403}}{9}.$$

(12) $1\cdot0\dot{3}=\dfrac{103-10}{90}=\dfrac{93}{90}=\dfrac{31}{30}$.

Let the measure of AP be 31, and that of AM be 30; then that of PM will be $\sqrt{961-900}$, or, $\sqrt{61}$.

$$\text{Then } \sin\theta=\frac{\sqrt{61}}{31}, \text{ and } \tan\theta=\frac{\sqrt{61}}{30}.$$

(13) Let the measure of PM be 99, and that of AP be 101; then that of AM will be $\sqrt{10201-9801}$, or, 20.

$$\text{Then } \cos\phi=\frac{20}{101}, \text{ and } \cot\phi=\frac{20}{99}.$$

(14) Let the measure of AM be 20, and that of AP be 101; then that of PM will be $\sqrt{10201-400}$, or, 99.

$$\text{Then } \sin\phi=\frac{99}{101}, \text{ and } \tan\phi=\frac{99}{20}.$$

(15) $\text{Cos}\theta=1-\text{versin}\theta=1-\dfrac{1}{13}=\dfrac{12}{13}$.

Let the measure of AM be 12, and that of AP be 13; then that of PM will be $\sqrt{169-144}$, or, 5.

$$\text{Then } \sin\theta=\frac{5}{13}, \text{ and } \sec\theta=\frac{13}{12}.$$

EXAMPLES—XIX. (p. 63).

(1) $\sin A=\dfrac{1}{\text{cosec}A}=\dfrac{1}{\sqrt{\text{cosec}^2A}}=\dfrac{1}{\sqrt{(1+\cot^2A)}}$.

(2) $\cos A=\dfrac{1}{\sec A}=\dfrac{1}{\sqrt{\sec^2A}}=\dfrac{1}{\sqrt{(1+\tan^2A)}}$.

(3) $\cos x=\dfrac{\cot x}{\text{cosec}x}=\dfrac{\cot x}{\sqrt{(\text{cosec}^2x)}}=\dfrac{\cot x}{\sqrt{(1+\cot^2x)}}$.

(4) $\tan x\,.\,\cos x=\sin x=\sqrt{(1-\cos^2x)}$.

(5) $\cos\phi=\dfrac{\cot\phi}{\text{cosec}\phi}=\dfrac{\sqrt{(\text{cosec}^2\phi-1)}}{\text{cosec}\phi}$.

(6) $$\tan\phi=\frac{\sin\phi}{\cos\phi}=\frac{\sqrt{(1-\cos^2\phi)}}{\cos\phi}=\sqrt{\left(\frac{1-\cos^2\phi}{\cos^2\phi}\right)}.$$

(7) $$\sin^2\alpha=1-\cos^2\alpha=(1+\cos\alpha)(1-\cos\alpha)=(1+\cos\alpha)\,.\,\text{versin}\,\alpha.$$

(8) $$\tan^2\alpha-\tan^2\beta=\frac{\sin^2\alpha}{\cos^2\alpha}-\frac{\sin^2\beta}{\cos^2\beta}=\frac{\sin^2\alpha\,.\,\cos^2\beta-\cos^2\alpha\,.\,\sin^2\beta}{\cos^2\alpha\,.\,\cos^2\beta}$$
$$=\frac{(1-\cos^2\alpha)\cos^2\beta-(1-\cos^2\beta)\cos^2\alpha}{\cos^2\alpha\,.\,\cos^2\beta}=\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\alpha\,.\,\cos^2\beta}.$$

(9) $$\cot^2\alpha-\cot^2\beta=\frac{\cos^2\alpha}{\sin^2\alpha}-\frac{\cos^2\beta}{\sin^2\beta}=\frac{\cos^2\alpha\,.\,\sin^2\beta-\cos^2\beta\,.\,\sin^2\alpha}{\sin^2\alpha\,.\,\sin^2\beta}$$
$$=\frac{(1-\sin^2\alpha)\sin^2\beta-(1-\sin^2\beta)\sin^2\alpha}{\sin^2\alpha\,.\,\sin^2\beta}=\frac{\sin^2\beta-\sin^2\alpha}{\sin^2\alpha\,.\,\sin^2\beta}.$$

(10) $$\sin^2\theta\,.\,\tan^2\theta+\cos^2\theta\,.\,\cot^2\theta=(1-\cos^2\theta)\,.\,\tan^2\theta+(1-\sin^2\theta)\,.\,\cot^2\theta$$
$$=\tan^2\theta-\sin^2\theta+\cot^2\theta-\cos^2\theta=\tan^2\theta+\cot^2\theta-(\sin^2\theta+\cos^2\theta)$$
$$=\tan^2\theta+\cot^2\theta-1.$$

(11) $$\sec^4\theta+\tan^4\theta=(1+\tan^2\theta)^2+\tan^4\theta=1+2\tan^2\theta+\tan^4\theta+\tan^4\theta$$
$$=1+2\tan^2\theta(1+\tan^2\theta)=1+2\tan^2\theta\,.\,\sec^2\theta.$$

(12) $$\text{cosec}\,\theta(\sec\theta-1)-\cot\theta(1-\cos\theta)=\frac{1}{\sin\theta.\cos\theta}-\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}+\frac{\cos^2\theta}{\sin\theta}$$
$$=\frac{1-\cos^2\theta}{\sin\theta\,.\,\cos\theta}-\frac{1-\cos^2\theta}{\sin\theta}=\frac{\sin^2\theta}{\sin\theta\,.\,\cos\theta}-\frac{\sin^2\theta}{\sin\theta}=\tan\theta-\sin\theta.$$

(13) $$\cot^2b+\tan^2b=(\text{cosec}^2b-1)+(\sec^2b-1)=\text{cosec}^2b+\sec^2b-2$$
$$=\frac{1}{\sin^2b}+\frac{1}{\cos^2b}-2=\frac{\cos^2b+\sin^2b}{\sin^2b\,.\,\cos^2b}-2.$$
$$=\frac{1}{\sin^2b\,.\,\cos^2b}-2=\text{cosec}^2b\,.\,\sec^2b-2.$$

(14) $$\cot^2A-\cos^2A=\frac{\cos^2A}{\sin^2A}-\cos^2A=\cos^2A\left(\frac{1}{\sin^2A}-1\right)$$
$$=\cos^2A\left(\frac{1-\sin^2A}{\sin^2A}\right)=\cos^2A\,.\,\frac{\cos^2A}{\sin^2A}=\cos^4A\,.\,\text{cosec}^2A.$$

(15) $$\tan^2\theta-\sin^2\theta=\frac{\sin^2\theta}{\cos^2\theta}-\sin^2\theta=\sin^2\theta\left(\frac{1}{\cos^2\theta}-1\right)$$
$$=\sin^2\theta\,.\,\frac{1-\cos^2\theta}{\cos^2\theta}=\sin^2\theta\,.\,\frac{\sin^2\theta}{\cos^2\theta}=\sin^4\theta\,.\,\sec^2\theta.$$

(16) $(\sec\theta - \operatorname{cosec}\theta)(1 + \cot\theta + \tan\theta) = \left(\frac{1}{\cos\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)$

$$= \frac{\sin\theta - \cos\theta}{\sin\theta\,.\,\cos\theta} \cdot \frac{\sin\theta\,.\,\cos\theta + 1}{\sin\theta\,.\,\cos\theta} = \frac{\sin^2\theta\,.\,\cos\theta + \sin\theta - \sin\theta\,.\,\cos^2\theta - \cos\theta}{\sin^2\theta\,.\,\cos^2\theta}$$

$$= \frac{(1 - \cos^2\theta)\cos\theta + \sin\theta - \sin\theta(1 - \sin^2\theta) - \cos\theta}{\sin^2\theta\,.\,\cos^2\theta}$$

$$= \frac{\cos\theta - \cos^3\theta + \sin\theta - \sin\theta + \sin^3\theta - \cos\theta}{\sin^2\theta\,.\,\cos^2\theta}$$

$$= \frac{\sin^3\theta - \cos^3\theta}{\sin^2\theta\,.\,\cos^2\theta} = \frac{\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta} = \frac{\sec^2\theta}{\operatorname{cosec}\theta} - \frac{\operatorname{cosec}^2\theta}{\sec\theta}.$$

(17) $\frac{\operatorname{cosec}\theta}{\sec\theta} + \frac{\sec\theta}{\operatorname{cosec}\theta} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\,.\,\cos\theta}$

$$= \frac{1}{\sin\theta\,.\,\cos\theta} = \sec\theta\,.\,\operatorname{cosec}\theta.$$

(18) $\cos\theta(\tan\theta + 2)(2\tan\theta + 1) = \cos\theta(2\tan^2\theta + 5\tan\theta + 2)$

$$\begin{aligned} &= 2\cos\theta(\tan^2\theta + 1) + 5\cos\theta\,.\,\tan\theta \\ &= 2\cos\theta\,.\,\sec^2\theta + 5\,.\,\sin\theta = 2\sec\theta + 5\sin\theta. \end{aligned}$$

(19) $\cos x(2\sec x + \tan x)(\sec x - 2\tan x)$

$$\begin{aligned} &= \cos x(2\sec^2 x - 3\sec x\,.\,\tan x - 2\tan^2 x) \\ &= 2\cos x(\sec^2 x - \tan^2 x) - 3\cos x\,.\,\sec x\,.\,\tan x \\ &= 2\cos x - 3\tan x. \end{aligned}$$

(20) $(\operatorname{cosec}\theta - \cot\theta)^2 = \operatorname{cosec}^2\theta - 2\operatorname{cosec}\theta\,.\,\cot\theta + \cot^2\theta$

$$\begin{aligned} &= \frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \\ &= \frac{1 - 2\cos\theta + \cos^2\theta}{\sin^2\theta} = \frac{(1 - \cos\theta)^2}{1 - \cos^2\theta} \\ &\qquad = \frac{(1 - \cos\theta)(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)} \\ &= \frac{1 - \cos\theta}{1 + \cos\theta}. \end{aligned}$$

(21) $\frac{\sec\theta\,.\,\cot\theta - \operatorname{cosec}\theta\,.\,\tan\theta}{\cos\theta - \sin\theta} = \frac{\frac{1}{\sin\theta} - \frac{1}{\cos\theta}}{\cos\theta - \sin\theta} = \frac{\frac{\cos\theta - \sin\theta}{\sin\theta\,.\,\cos\theta}}{\cos\theta - \sin\theta}$

$$= \frac{1}{\sin\theta\,.\,\cos\theta} = \operatorname{cosec}\theta\,.\,\sec\theta.$$

(22) $\sec\theta + \operatorname{cosec}\theta \,.\, \tan^3\theta(1 + \operatorname{cosec}^2\theta) = \dfrac{1}{\cos\theta} + \dfrac{\sin^2\theta}{\cos^3\theta} + \dfrac{1}{\cos^3\theta}$

$$= \frac{\cos^2\theta + \sin^2\theta + 1}{\cos^3\theta} = \frac{2}{\cos^3\theta} = 2\sec^3\theta.$$

(23) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$$= \sin^2\theta + \frac{2\sin\theta}{\cos\theta} + \frac{1}{\cos^2\theta} + \cos^2\theta + \frac{2\cos\theta}{\sin\theta} + \frac{1}{\sin^2\theta}$$

$$= (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + \left(\frac{2\sin\theta}{\cos\theta} + \frac{2\cos\theta}{\sin\theta}\right)$$

$$= 1 + \frac{1}{\sin^2\theta \,.\, \cos^2\theta} + \frac{2}{\sin\theta \,.\, \cos\theta} = \left(1 + \frac{1}{\sin\theta . \cos\theta}\right)^2 = (1 + \sec\theta . \operatorname{cosec}\theta)^2.$$

(24) $\dfrac{1 + (\operatorname{cosec}\theta \,.\, \tan\phi)^2}{1 + (\operatorname{cosec}a \,.\, \tan\phi)^2} = \dfrac{1 + \dfrac{\sin^2\phi}{\sin^2\theta \,.\, \cos^2\phi}}{1 + \dfrac{\sin^2\phi}{\sin^2 a \,.\, \cos^2\phi}} = \dfrac{\sin^2\theta \,.\, \cos^2\phi + \sin^2\phi}{\sin^2 a \,.\, \cos^2\phi + \sin^2\phi} \cdot \dfrac{\sin^2 a}{\sin^2\theta}$

$$= \frac{\sin^2\theta(1 - \sin^2\phi) + \sin^2\phi}{\sin^2 a(1 - \sin^2\phi) + \sin^2\phi} \cdot \frac{\sin^2 a}{\sin^2\theta} = \frac{\sin^2\theta - \sin^2\theta \,.\, \sin^2\phi + \sin^2\phi}{\sin^2 a - \sin^2 a \,.\, \sin^2\phi + \sin^2\phi} \cdot \frac{\sin^2 a}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \sin^2\phi \,.\, \cos^2\theta}{\sin^2 a + \sin^2\phi \,.\, \cos^2 a} \cdot \frac{\sin^2 a}{\sin^2\theta}$$

$$= \frac{1 + \sin^2\phi \,.\, \cot^2\theta}{1 + \sin^2\phi \,.\, \cot^2 a} = \frac{1 + (\cot\theta \,.\, \sin\phi)^2}{1 + (\cot a \,.\, \sin\phi)^2}.$$

(25) $(3 - 4\sin^2 A)(1 - 3\tan^2 A) = (3 - 4\sin^2 A)\left(1 - \dfrac{3\sin^2 A}{\cos^2 A}\right)$

$$= (3 - 4\sin^2 A)\left(\frac{\cos^2 A - 3\sin^2 A}{\cos^2 A}\right)$$

$$= (3 - 4\sin^2 A)\left(\frac{\cos^2 A - 3(1 - \cos^2 A)}{\cos^2 A}\right)$$

$$= \frac{3 - 4\sin^2 A}{\cos^2 A} \cdot (4\cos^2 A - 3)$$

$$= \frac{3\cos^2 A + 3\sin^2 A - 4\sin^2 A}{\cos^2 A}(4\cos^2 A - 3)$$

$$= \frac{3\cos^2 A - \sin^2 A}{\cos^2 A}(4\cos^2 A - 3)$$

$$= (3 - \tan^2 A)(4\cos^2 A - 3).$$

Examples—XX. (p. 65).

1. (1) $90^\circ - (24^\circ.\,14'.\,42'') = 65^\circ.\,45'.\,18''$.

(2) $90^\circ - (43^\circ.\,2'.\,57'') = 46^\circ.\,57'.\,3''$.

(3) $90^\circ - (64^\circ.\,0'.\,14'') = 25^\circ.\,59'.\,46''$.

(4) $90^\circ - (82^\circ.\,4'.\,15'') = 7^\circ.\,55'.\,45''$.

(5) $90^\circ - (125^\circ.\,15'.\,42'') = -(35^\circ.\,15'.\,42'')$.

(6) $90^\circ - (178^\circ.\,27'.\,34'') = -(88^\circ.\,27'.\,34'')$.

(7) $90^\circ - 195^\circ = -105^\circ$.

(8) $90^\circ - 254^\circ = -164^\circ$.

(9) $90^\circ - (-25^\circ) = 90^\circ + 25^\circ = 115^\circ$.

(10) $90^\circ - (-245^\circ) = 90^\circ + 245^\circ = 335^\circ$.

2. (1) $100^g - (32^g.\,23'.\,24'') = 67^g.\,76'.\,76''$.

(2) $100^g - (95^g.\,3'.\,75'') = 4^g.\,96'.\,25''$.

(3) $100^g - (46^g.\,0'.\,84'') = 53^g.\,99'.\,16''$.

(4) $100^g - (2^g.\,5'.\,4'') = 97^g.\,94'.\,96''$.

(5) $100^g - (135^g.\,2'.\,5'') = -(35^g.\,2'.\,5'')$.

(6) $100^g - (169^g.\,0'.\,3'') = -(69^g.\,0'.\,3'')$.

(7) $100^g - 243^g = -143^g$.

(8) $100^g - 357^g = -257^g$.

(9) $100^g - (-35^g) = 100^g + 35^g = 135^g$.

(10) $100^g - (-245^g) = 100^g + 245^g = 345^g$.

3. (1) $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. (2) $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$. (3) $\frac{\pi}{2} - \frac{3\pi}{5} = -\frac{\pi}{10}$.

(4) $\frac{\pi}{2} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$. (5) $\frac{\pi}{2} - \left(-\frac{3\pi}{4}\right) = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4}$.

EXAMPLES—XXI. (p. 68).

1. (1) $180^\circ - (34^\circ.\,12'.\,49'') = 145^\circ.\,47'.\,11''.$

(2) $180^\circ - (132^\circ.\,24'.\,47'') = 47^\circ.\,35'.\,13''.$

(3) $180^\circ - (146^\circ.\,0'.\,41'') = 33^\circ.\,59'.\,19''.$

(4) $180^\circ - (28^\circ.\,15'.\,4'') = 151^\circ.\,44'.\,56''.$

(5) $180^\circ - (179^\circ.\,59'.\,59'') = \quad 1''.$

(6) $180^\circ - (100^\circ.\,49'.\,53'') = 79^\circ.\,10'.\,7''.$

(7) $180^\circ - 245^\circ \quad = -65^\circ.$

(8) $180^\circ - (437^\circ.\,3'.\,4'') = -(257^\circ.\,3'.\,4'').$

(9) $180^\circ - (-49^\circ) = 180^\circ + 49^\circ = 229^\circ.$

(10) $180^\circ - (-355^\circ) = 180^\circ + 355^\circ = 535^\circ.$

2. (1) $200^g - (132^g.\,32'.\,42'') = 67^g.\,67'.\,58''.$

(2) $200^g - (195^g.\,2'.\,57'') = 4^g.\,97'.\,43''.$

(3) $200^g - (3^g.\,97'.\,98'') = 196^g.\,2'.\,2''.$

(4) $200^g - (65^g.\,12'.\,8'') = 134^g.\,87'.\,92''.$

(5) $200^g - (154^g.\,3'.\,6'') = 45^g.\,96'.\,94''.$

(6) $200^g - (174^g.\,0'.\,4'') = 25^g.\,99'.\,96''.$

(7) $200^g - 275^g = -75^g.$

(8) $200^g - (527^g.\,2'.\,14'') = -(327^g.\,2'.\,14'').$

(9) $200^g - (-35^g) = 200^g + 35^g = 235^g.$

(10) $200^g - (-325^g) = 200^g + 325^g = 525^g.$

3. (1) $\pi - \frac{\pi}{2} = \frac{\pi}{2}.$ (2) $\pi - \frac{\pi}{3} = \frac{2\pi}{3}.$ (3) $\pi - \frac{4\pi}{5} = \frac{\pi}{5}.$

(4) $\pi - \left(-\frac{\pi}{4}\right) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$ (5) $\pi - \left(-\frac{3\pi}{4}\right) = \pi + \frac{3\pi}{4} = \frac{7\pi}{4}.$

4. Let θ be the circular measure of the angle.

Then $\frac{\pi}{2} - \theta$ is the complement of θ;

and $\pi - \left(\frac{\pi}{2} - \theta\right)$, or, $\frac{\pi}{2} + \theta$ is the supplement of the complement of θ.

Again $\pi - \theta$ is the supplement of θ,

and $\frac{\pi}{2} - (\pi - \theta)$, or, $\theta - \frac{\pi}{2}$ is the complement of the supplement of θ;

$$\therefore \text{difference} = \frac{\pi}{2} + \theta - \left(\theta - \frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

EXAMPLES—XXII. (p. 72).

1. (1) Take the construction and notation of Art. 101.

$$\text{Then } \sec(180° - A) = \sec EOP' = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec A.$$

(2) Take the construction of Art. 102, and let $\angle EOP = \theta$.

$$\text{Then } \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \frac{OP'}{P'M'} = \frac{OP}{OM} = \sec\theta.$$

(3) Take the construction and notation of Art. 103.

$$\text{Then } \tan(180° + A) = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \frac{PM}{OM} = \tan A.$$

(4) Take the construction of Art. 103, and let $\angle EOP = \theta$.

$$\text{Then } \sec(\pi + \theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta.$$

(5) Take the construction of Art. 104, and let $\angle EOP = \theta$.

$$\text{Then } \tan(-\theta) = \frac{MP'}{MO} = \frac{-MP}{MO} = -\tan\theta.$$

(6) Take the construction of Art. 104, and let $\angle EOP = \theta$.

$$\text{Then } \cot(2\pi - \theta) = \cot EOP' = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\theta.$$

2. (1) Take the construction of Art. 102, and let $\angle EOP = B$.

$$\text{Then } \operatorname{cosec}(90^\circ + B) = \operatorname{cosec} EOP' = \frac{OP'}{P'M'} = \frac{OP}{OM} = \sec B = \frac{\operatorname{cosec} B}{\sqrt{\operatorname{cosec}^2 B - 1}}.$$

(Ex. XVII. 2.)

(2) Take the construction of Art. 103, and let $\angle EOP = \phi$.

$$\text{Then } \operatorname{cosec}(\pi + \phi) = \operatorname{cosec} EOP' = \frac{OP'}{P'M'} = \frac{OP}{-PM} = -\operatorname{cosec}\phi.$$

3. (1) Take the construction of Art. 102, and let $\angle EOP = A$.

$$\text{Then } \sec(90^\circ + A) = \sec EOP' = \frac{OP'}{OM'} = \frac{OP}{-PM} = -\operatorname{cosec} A = -\frac{\sec A}{\sqrt{\sec^2 A - 1}}.$$

(Ex. XVII. 3.)

(2) Take the construction of Art. 99, and let $\angle EOP = \theta$.

$$\text{Then } \sec\left(\frac{\pi}{2} - \theta\right) = \sec EOP' = \frac{OP'}{OM'} = \frac{OP}{PM} = \operatorname{cosec}\theta = \frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}.$$

(Ex. XVII. 3.)

EXAMPLES—XXIII. (p. 72).

(1) $\sin 120^\circ = \sin(180^\circ - 120^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$.

(2) $\cos 120^\circ = -\cos(180^\circ - 120^\circ) = -\cos 60^\circ = -\frac{1}{2}$.

(3) $\sin 135^\circ = \sin(180^\circ - 135^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$.

(4) $\cos 135^\circ = -\cos(180^\circ - 135^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$.

(5) $\sin 150^\circ = \sin(180^\circ - 150^\circ) = \sin 30^\circ = \frac{1}{2}$.

(6) $\cos 150^\circ = -\cos(180^\circ - 150^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$.

(7) $\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$.

(8) $\sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$.

(9) $\tan 300^\circ = \tan(360^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$.

(10) $\text{cosec}\,300^\circ = \text{cosec}(360^\circ - 60^\circ) = -\text{cosec}\,60^\circ = -\frac{2}{\sqrt{3}}$.

(11) $\sec 315^\circ = \sec(360^\circ - 45^\circ) = \sec 45^\circ = \sqrt{2}$.

(12) $\cot 330^\circ = \cot(360^\circ - 30^\circ) = -\cot 30^\circ = -\sqrt{3}$.

EXAMPLES—XXIV. (p. 75).

(1) $\sin\theta + \cos\theta = 0$,

$$\sin\theta = -\cos\theta,$$
$$\sin^2\theta = \cos^2\theta,$$
$$\sin^2\theta = 1 - \sin^2\theta,$$
$$2\sin^2\theta = 1.$$

Hence $\sin\theta = \pm\frac{1}{\sqrt{2}}$, and $\therefore\ \theta = 45^\circ$ or -45°.

The latter of these values must be taken, because $\sin\theta$ and $\cos\theta$ must have different signs to satisfy the equation.

(2) $\sin\theta - \cos\theta = 0$,

$$\sin\theta = \cos\theta,$$

and, as in Example (1), $\theta = 45^\circ$ or -45°.

The former of these values must be taken, because $\sin\theta$ and $\cos\theta$ must have the same sign to satisfy the equation.

(3) $\sin\theta = \tan\theta$,

$\sin\theta = \frac{\sin\theta}{\cos\theta}$, and, dividing by $\sin\theta$,

$1 = \frac{1}{\cos\theta}$, or, $\cos\theta = 1$, and $\therefore\ \theta = 0^\circ$.

(4) $\cos\theta = \cot\theta$,

$\cos\theta = \frac{\cos\theta}{\sin\theta}$, and, dividing by $\cos\theta$,

$1 = \frac{1}{\sin\theta}$, or, $\sin\theta = 1$, and $\therefore\ \theta = 90^\circ$.

(5) $2\sin\theta=\tan\theta,$

$$2\sin\theta=\frac{\sin\theta}{\cos\theta}, \text{ or, } 2\cos\theta=1, \text{ or, } \cos\theta=\frac{1}{2}, \text{ and, } \therefore, \theta=60^\circ.$$

Also, since we divided by $\sin\theta$, one value of θ to satisfy the original equation is given by $\sin\theta=0$, or, $\theta=0^\circ$.

(6)

$$3\sin\theta=2\cos^2\theta,$$
$$3\sin\theta=2(1-\sin^2\theta),$$
$$2\sin^2\theta+3\sin\theta=2,$$
$$\sin^2\theta+\frac{3}{2}\sin\theta=1.$$
$$\sin^2\theta+\frac{3}{2}\sin\theta+\frac{9}{16}=\frac{25}{16}.$$
$$\sin\theta+\frac{3}{4}=\pm\frac{5}{4}.$$

Hence $\sin\theta=\frac{1}{2}$, or, -2.

The second value is inadmissible

$$\therefore\ \sin\theta=\frac{1}{2}, \text{ or, } \theta=30^\circ.$$

(7)

$$\sin\theta+\cos^2\theta\,.\,\operatorname{cosec}\theta=2,$$
$$\sin\theta+\frac{\cos^2\theta}{\sin\theta}=2,$$
$$\sin^2\theta+\cos^2\theta=2\sin\theta,$$
$$1=2\sin\theta.$$

Hence $\sin\theta=\frac{1}{2}$, or, $\theta=30^\circ$.

(8)

$$\tan\theta=4-3\cot\theta,$$
$$\tan\theta+3\cot\theta=4,$$
$$\tan\theta+\frac{3}{\tan\theta}=4,$$
$$\tan^2\theta+3=4\tan\theta,$$
$$\tan^2\theta-4\tan\theta=-3,$$
$$\tan^2\theta-4\tan\theta+4=1,$$
$$\tan\theta-2=\pm1.$$

Hence $\tan\theta=3$ or 1, and the latter of these values of $\tan\theta$ enables us to say that *one* value of θ is 45°.

(9)
$$4\sec^2\theta - 7\tan^2\theta = 3,$$
$$4(1+\tan^2\theta) - 7\tan^2\theta = 3,$$
$$4 - 3\tan^2\theta = 3,$$
$$\tan^2\theta = \frac{1}{3}, \text{ or, } \tan\theta = \frac{1}{\sqrt{3}}, \text{ and } \therefore\ \theta = 30^\circ.$$

(10)
$$\cos\theta \,.\, \operatorname{cosec}\theta + \sin\theta \,.\, \sec\theta = \frac{4}{\sqrt{3}},$$
$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{4}{\sqrt{3}},$$
$$\frac{\cos^2\theta + \sin^2\theta}{\sin\theta \,.\, \cos\theta} = \frac{4}{\sqrt{3}},$$
$$\sqrt{3} = 4\sin\theta \,.\, \cos\theta,$$
$$3 = 16\sin^2\theta(1 - \sin^2\theta),$$
$$16\sin^4\theta - 16\sin^2\theta = -3,$$
$$\sin^4\theta - \sin^2\theta = -\frac{3}{16}.$$
$$\text{Hence } \sin^2\theta = \frac{3}{4} \text{ or } \frac{1}{4},$$
$$\text{and } \therefore\ \sin\theta = \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2}, \text{ and } \theta = 60^\circ \text{ or } 30^\circ.$$

(11)
$$3\sin^2\theta - \cos^2\theta + (\sqrt{3}+1)(1 - 2\sin\theta) = 0,$$
$$3\sin^2\theta - (1 - \sin^2\theta) + \sqrt{3} + 1 - 2\sqrt{3}\sin\theta - 2\sin\theta = 0,$$
$$4\sin^2\theta - 2(\sqrt{3}+1)\sin\theta = -\sqrt{3},$$
$$\sin^2\theta - \frac{\sqrt{3}+1}{2} \cdot \sin\theta = -\frac{\sqrt{3}}{4},$$
$$\sin^2\theta - \frac{\sqrt{3}+1}{2}\sin\theta + \frac{4+2\sqrt{3}}{16} = \frac{4+2\sqrt{3}}{16} - \frac{\sqrt{3}}{4} = \frac{4-2\sqrt{3}}{16},$$
$$\sin\theta - \frac{\sqrt{3}+1}{4} = \pm\frac{\sqrt{3}-1}{4}.$$
$$\text{Hence } \sin\theta = \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2}, \text{ and } \theta = 60^\circ \text{ or } 30^\circ.$$

(12)
$$3\cos^2\theta - \sin^2\theta + (\sqrt{3}+1)(1 - 2\cos\theta) = 0,$$
$$3\cos^2\theta - (1 - \cos^2\theta) + \sqrt{3} + 1 - 2\sqrt{3}\cos\theta - 2\cos\theta = 0,$$
$$4\cos^2\theta - 2(\sqrt{3}+1)\cos\theta = -\sqrt{3}.$$

Hence, by the same process as in Example (11),
$$\cos\theta = \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2}, \text{ and } \theta = 30^\circ \text{ or } 60^\circ.$$

(13)
$$\sec\theta\,.\,\operatorname{cosec}\theta+2\cot\theta=4,$$
$$\frac{1}{\cos\theta\,.\,\sin\theta}+\frac{2\cos\theta}{\sin\theta}=4,$$
$$1+2\cos^2\theta=4\sin\theta\,.\,\cos\theta,$$
$$1+4\cos^2\theta+4\cos^4\theta=16\sin^2\theta\,.\,\cos^2\theta,$$
$$1+4\cos^2\theta+4\cos^4\theta=16\cos^2\theta-16\cos^4\theta,$$
$$20\cos^4\theta-12\cos^2\theta=-1,$$
$$\cos^4\theta-\frac{3}{5}\cos^2\theta=-\frac{1}{20}.$$

Hence $\cos^2\theta=\frac{1}{2}$, and $\cos\theta=\frac{1}{\sqrt{2}}$, and $\theta=45°$

(14)
$$\sin\theta+\cos\theta=\sqrt{2}, \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$\sin^2\theta+2\sin\theta\,.\,\cos\theta+\cos^2\theta=2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$2\sin\theta\,.\,\cos\theta=1,$$

$4\sin\theta\,.\,\cos\theta=2$, and, subtracting this from (2),

$$\sin^2\theta-2\sin\theta\,.\,\cos\theta+\cos^2\theta=0,$$

$\sin\theta-\cos\theta=0$, and, adding this to (1),

$2\sin\theta=\sqrt{2}$, $\therefore$ $\sin\theta=\frac{1}{\sqrt{2}}$, and $\theta=45°$.

(15)
$$\cot^2\theta+4\cos^2\theta=6,$$
$$\cos^2\theta+4\cos^2\theta\,.\,\sin^2\theta=6\sin^2\theta,$$
$$\cos^2\theta+4\cos^2\theta-4\cos^4\theta=6-6\cos^2\theta,$$
$$\cos^4\theta-\frac{11}{4}\cos^2\theta=-\frac{3}{2}.$$

Hence $\cos^2\theta=\frac{3}{4}$ or 2.

The latter value is inadmissible, and we must have

$\cos^2\theta=\frac{3}{4}$, or, $\cos\theta=\frac{\sqrt{3}}{2}$, and $\theta=30°$.

(16)
$$\tan\theta+\cot\theta=2.$$
$$\tan\theta+\frac{1}{\tan\theta}=2,$$
$$\tan^2\theta-2\tan\theta=-1\,;$$

$\therefore$ $\tan\theta=1$, and $\theta=45°$.

(17) $$\sin\theta-\cos\theta=\sqrt{2}, \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$\sin^2\theta-2\sin\theta.\cos\theta+\cos^2\theta=2, \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$-2\sin\theta.\cos\theta=1,$$

$$4\sin\theta.\cos\theta=-2, \text{ and adding this to (2)}$$

$$\sin^2\theta+2\sin\theta.\cos\theta+\cos^2\theta=0,$$

$$\sin\theta+\cos\theta=0, \text{ and adding this to (1)}$$

$$2\sin\theta=\sqrt{2}, \text{or}, \sin\theta=\frac{1}{\sqrt{2}}, \text{ and } \theta=45^\circ \text{ or } 135^\circ.$$

Now $\cos\theta$ has to be of the same numerical value as $\sin\theta$, but with a different sign, and hence 45° is an inadmissible value of θ;

$$\therefore\ \theta=135^\circ.$$

(18) $$\sin\theta+\cos\theta=2\sqrt{2}.\sin\theta.\cos\theta,$$

$$\sin^2\theta+2\sin\theta.\cos\theta+\cos^2\theta=8\sin^2\theta.\cos^2\theta,$$

$$8\sin^2\theta.\cos^2\theta-2\sin\theta.\cos\theta=1,$$

$$\sin^2\theta.\cos^2\theta-\frac{1}{4}.\sin\theta.\cos\theta=\frac{1}{8},$$

$$\sin^2\theta.\cos^2\theta-\frac{1}{4}\sin\theta.\cos\theta+\frac{1}{64}=\frac{9}{64};$$

$$\therefore \sin\theta.\cos\theta=\frac{1}{2} \text{ or } -\frac{1}{4}.$$

Taking the former of these values, we get

$$\sin^2\theta\,(1-\sin^2\theta)=\frac{1}{4}.$$

Whence $\sin^2\theta=\frac{1}{2}$, or, $\sin\theta=\frac{1}{\sqrt{2}}$, and $\theta=45^\circ$.

(19) $$\sqrt{3}.\sin\theta=\sqrt{3}-\cos\theta,$$

$$3\sin^2\theta=3-2\sqrt{3}.\cos\theta+\cos^2\theta,$$

$$3-3\cos^2\theta=3-2\sqrt{3}\cos\theta+\cos^2\theta,$$

$$4\cos^2\theta=2\sqrt{3}.\cos\theta.$$

Dividing by $\cos\theta$, we get

$$4\cos\theta=2\sqrt{3}, \text{ or}, \cos\theta=0.$$

Hence $\cos\theta=\frac{\sqrt{3}}{2}$, ór, $\cos\theta=0$;

$$\therefore\ \theta=30^\circ \text{ or } 90^\circ.$$

(20)
$$\tan^2\theta + 4\sin^2\theta = 3,$$
$$\sin^2\theta + 4\sin^2\theta \,.\, \cos^2\theta = 3\cos^2\theta,$$
$$1 - \cos^2\theta + 4\cos^2\theta - 4\cos^4\theta = 3\cos^2\theta,$$
$$4\cos^4\theta = 1\,;$$

$\therefore$ one value of $\cos\theta$ is $\frac{1}{\sqrt{2}}$, or, $\theta = 45°$.

EXAMPLES—XXV. (p. 81).

(1) $\sin 480° = \sin(360° + 120°) = \sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$.

(2) $\cos 480° = \cos(360° + 120°) = \cos 120° = -\cos 60° = -\frac{1}{2}$.

(3) $\sin 495° = \sin(360° + 135°) = \sin 135° = \sin 45° = \frac{1}{\sqrt{2}}$.

(4) $\cos 495° = \cos(360° + 135°) = \cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}$.

(5) $\sin 870° = \sin(720° + 150°) = \sin 150° = \sin 30° = \frac{1}{2}$.

(6) $\cos 870° = \cos(720° + 150°) = \cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$.

(7) $\sin 945° = \sin(720° + 225°) = \sin 225° = -\sin 45° = -\frac{1}{\sqrt{2}}$.

(8) $\sin 960° = \sin(720° + 240°) = \sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$.

(9) $\tan 1020° = \tan(720° + 300°) = \tan 300° = -\tan 60° = -\sqrt{3}$.

(10) $\operatorname{cosec} 1380° = \operatorname{cosec}(1080° + 300°) = \operatorname{cosec} 300° = -\operatorname{cosec} 60° = -\frac{2}{\sqrt{3}}$.

(11) $\sec 1395° = \sec(1080° + 315°) = \sec 315° = \sec 45° = \sqrt{2}$.

(12) $\cot 1410° = \cot(1080° + 330°) = \cot 330° = -\cot 30° = -\sqrt{3}$.

(13) $\cos 420° = \cos(360° + 60°) = \cos 60° = \frac{1}{2}$.

(14) $\sec 750° = \sec(720° + 30°) = \sec 30° = \frac{2}{\sqrt{3}}$.

(15) $\tan 945° = \tan(720° + 225°) = \tan 225° = \tan 45° = 1$.

(16) $\sin 1200° = \sin(1080° + 120°) = \sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$.

(17) $\sin 1485° = \sin(1440° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}$.

(18) $\cos 1470° = \cos(1440° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$.

(19) $\sin 7\pi = \sin(6\pi + \pi) = \sin \pi = 0$.

(20) $\sec 8\pi = \sec 2\pi = 1$.

(21) $\operatorname{cosec} 930° = \operatorname{cosec}(720° + 210°) = \operatorname{cosec} 210° = -\operatorname{cosec} 30° = -2$.

(22) $\cot 1140° = \cot(1080° + 60°) = \cot 60° = \frac{1}{\sqrt{3}}$.

(23) $\tan 1305° = \tan(1080° + 225°) = \tan 225° = \tan 45° = 1$.

(24) $\operatorname{cosec} 1740° = \operatorname{cosec}(1440° + 300°) = \operatorname{cosec} 300° = -\operatorname{cosec} 60° = -\frac{2}{\sqrt{3}}$.

(25) $\sin(-240°) = -\sin 240° = -\sin(-60°) = \sin 60° = \frac{\sqrt{3}}{2}$.

(26) $\cot(-675°) = \cot(720° - 675°) = \cot 45° = 1$.

(27) $\sec(-135°) = -\sec(180° - 135°) = -\sec 45° = -\sqrt{2}$.

(28) $\tan(-225°) = \tan(360° - 225°) = \tan 135° = -\tan 45° = -1$.

(29) $\operatorname{cosec}(-690°) = \operatorname{cosec}(720° - 690°) = \operatorname{cosec} 30° = 2$.

(30) $\cos(-120°) = \cos 120° = -\cos 60° = -\frac{1}{2}$.

EXAMPLES—XXVI. (p. 82).

(1) $\sin\theta = 1$; $\therefore$ one value of θ is $\frac{\pi}{2}$;

$\therefore$ general value of θ is $n\pi + (-1)^n . \frac{\pi}{2}$.

(2) $\cos\theta = 1$; $\therefore$ one value of θ is 0;

$\therefore$ general value of θ is $2n\pi$.

(3) $\sin\theta = \frac{1}{\sqrt{2}}$; $\therefore$ one value of θ is $\frac{\pi}{4}$;

$\therefore$ general value of θ is $n\pi + (-1)^n . \frac{\pi}{4}$.

(4) $\tan\theta = \sqrt{3}$; $\therefore$ one value of θ is $\frac{\pi}{3}$;

$\therefore$ general value of θ is $n\pi + \frac{\pi}{3}$.

(5)
$$3\sin\theta = 2\cos^2\theta$$
$$3\sin\theta = 2(1 - \sin^2\theta)$$
$$\sin^2\theta + \frac{3}{2}\sin\theta = 1$$
$$\left(\sin\theta + \frac{3}{4}\right)^2 = \pm\frac{5}{4}, \text{ or, } \sin\theta = \frac{1}{2} \text{ or } -2$$

$\therefore$ least positive value of θ is $\frac{\pi}{6}$;

$\therefore$ general value of θ is $n\pi + (-1)^n . \frac{\pi}{6}$.

(6) $2\sin\theta = \tan\theta$,

$$2\sin\theta = \frac{\sin\theta}{\cos\theta};$$

$\therefore \sin\theta = 0$, or, $\cos\theta = \frac{1}{2}$;

$\therefore \theta = 0$, or, $\theta = \frac{\pi}{3}$;

$\therefore$ general value of θ is $n\pi$ or $2n\pi \pm \frac{\pi}{3}$.

(7) $$\tan^2\theta+4\sin^2\theta=3,$$

$$\sin^2\theta+4\sin^2\theta\,.\cos^2\theta=3\cos^2\theta,$$

$$\sin^2\theta+4\sin^2\theta-4\sin^4\theta=3-3\sin^2\theta,$$

$$4\sin^4\theta-8\sin^2\theta=-3,$$

$$\sin^4\theta-2\sin^2\theta+1=\frac{1}{4},$$

$$\sin^2\theta-1=\pm\frac{1}{2}.$$

Hence $\sin\theta=\pm\sqrt{\frac{3}{2}}$ or $\pm\frac{1}{\sqrt{2}}$;

∴ least positive value of θ is $\frac{\pi}{4}$;

∴ general value of θ is $n\pi+(-1)^n.\frac{\pi}{4}$.

(8) $$\cos^2=\sin^2\theta,$$

$$\cos^2\theta=1-\cos^2\theta,$$

$2\cos^2\theta=1$, and ∴ $\cos\theta=\pm\frac{1}{\sqrt{2}}$;

∴ the least positive values of θ are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$;

∴ the general value of θ is $2n\pi\pm\frac{\pi}{4}$ or $2n\pi\pm\frac{3\pi}{4}$.

(9) $$\tan\theta=4-3\cot\theta,$$

$$\tan\theta+\frac{3}{\tan\theta}=4,$$

$$\tan^2\theta-4\tan\theta=-3,$$

$$\tan\theta=3 \text{ or } 1;$$

∴ the least positive value of θ is $\frac{\pi}{4}$;

∴ general value of θ is $n\pi+\frac{\pi}{4}$.

$$(10)\ \sec^2\theta - \frac{5}{2}\sec\theta + 1 = 0,$$
$$\sec^2\theta - \frac{5}{2}\sec\theta + \frac{25}{16} = \frac{9}{16},$$
$$\sec\theta - \frac{5}{4} = \pm\frac{3}{4};$$
$$\therefore\ \sec\theta = 2 \text{ or } \frac{1}{2}.$$

Taking the value 2 for $\sec\theta$ (the other value being impossible) the general value of θ is $2n\pi \pm \frac{\pi}{3}$.

EXAMPLES—XXVII. (p. 87).

(1)
$$\begin{aligned}\sin(A+B).\sin(A-B) &= (\sin A.\cos B + \cos A.\sin B).(\sin A.\cos B - \cos A.\sin B)\\ &= \sin^2 A.\cos^2 B - \cos^2 A.\sin^2 B\\ &= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B\\ &= \sin^2 A - \sin^2 B.\end{aligned}$$

(2)
$$\begin{aligned}\sin(a+\beta).\sin(a-\beta) &= \sin a.\cos\beta + \cos a.\sin\beta)(\sin a.\cos\beta - \cos a.\sin\beta)\\ &= \sin^2 a.\cos^2\beta - \cos^2 a.\sin^2\beta\\ &= (1-\cos^2 a)\cos^2\beta - \cos^2 a(1-\cos^2\beta)\\ &= \cos^2\beta - \cos^2 a.\end{aligned}$$

(3)
$$\begin{aligned}\cos(A+B).\cos(A-B) &= (\cos A.\cos B - \sin A.\sin B)(\cos B.\cos B + \sin A.\sin B)\\ &= \cos^2 A.\cos^2 B - \sin^2 A.\sin^2 B\\ &= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B\\ &= \cos^2 A - \sin^2 B.\end{aligned}$$

(4)
$$\begin{aligned}\cos(a+\beta).\cos(a-\beta) &= (\cos a.\cos\beta - \sin a.\sin\beta)(\cos a.\cos\beta + \sin a.\sin\beta)\\ &= \cos^2 a.\cos^2\beta - \sin^2 a.\sin^2\beta\\ &= (1-\sin^2 a)\cos^2\beta - \sin^2 a(1-\cos^2\beta)\\ &= \cos^2\beta - \sin^2 a.\end{aligned}$$

(5)

$$\begin{aligned}
2\sin(x+y).\cos(x-y) &= 2(\sin x.\cos y+\cos x.\sin y).(\cos x.\cos y+\sin x.\sin y)\\
&= 2\{\sin x.\cos x.\cos^2 y+\sin^2 x.\cos y.\sin y+\cos^2 x.\sin y.\cos y+\\
&\qquad\qquad \sin x.\cos x.\sin^2 y\}\\
&= 2\{\sin x.\cos x(\cos^2 y+\sin^2 y)+\sin y.\cos y(\sin^2 x+\cos^2 x)\}\\
&= 2\{\sin x.\cos x+\sin y.\cos y\}\\
&= (\sin x.\cos x+\cos x.\sin x)+(\sin y.\cos y+\cos y.\sin y)\\
&= \sin(x+x)+\sin(y+y)\\
&= \sin 2x+\sin 2y
\end{aligned}$$

(6)

$$\begin{aligned}
2\cos(x+y).\sin(x-y) &= 2(\cos x.\cos y-\sin x.\sin y).(\sin x.\cos y-\cos x.\sin y)\\
&= 2\{\sin x.\cos x.\cos^2 y-\sin y.\cos y.\cos^2 x-\sin y.\cos y.\sin^2 x+\\
&\qquad\qquad \sin x.\cos x.\sin^2 y\}\\
&= 2\{\sin x.\cos x.(\cos^2 y+\sin^2 y)-\sin y.\cos y.(\cos^2 x+\sin^2 x)\}\\
&= 2\{\sin x.\cos x-\sin y.\cos y\}\\
&= (\sin x.\cos x+\cos x.\sin x)-(\sin y.\cos y+\cos y.\sin y)\\
&= \sin 2x-\sin 2y.
\end{aligned}$$

(7)
$$\begin{aligned}
\tan A+\tan B &= \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}.\\
&= \frac{\sin A.\cos B+\cos A.\sin B}{\cos A.\cos B}\\
&= \frac{\sin(A+B)}{\cos A.\cos B}.
\end{aligned}$$

(8)
$$\begin{aligned}
\tan\alpha-\tan\beta &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}\\
&= \frac{\sin\alpha.\cos\beta-\cos\alpha.\sin\beta}{\cos\alpha.\cos\beta}\\
&= \frac{\sin(\alpha-\beta)}{\cos\alpha.\cos\beta}.
\end{aligned}$$

EXAMPLES—XXVIII. (p. 88).

(1) $$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45° . \cos 30° - \cos 45° . \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.\end{aligned}$$

(2) $$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° . \cos 30° - \sin 45° . \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.\end{aligned}$$

(3) $$\begin{aligned}\tan 75° &= \tan(45° + 30°)\\ &= \sin(45° + 30°) \div \cos(45° + 30°)\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{3-1} = 2+\sqrt{3}.\end{aligned}$$

(4) $$\begin{aligned}\cot 75° &= \cos 75° \div \sin 75°\\ &= \cos(45° + 30°) \div \sin(45° + 30°)\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \div \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}\\ &= \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{3-1} = 2-\sqrt{3}.\end{aligned}$$

(5) If $\sin\alpha = \frac{1}{3}$, $\cos\alpha = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$.

If $\sin\beta = \frac{2}{3}$, $\cos\beta = \frac{\sqrt{5}}{3}$;

$$\therefore \sin(\alpha+\beta) = \frac{1}{3} \cdot \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3} \cdot \frac{2}{3} = \frac{\sqrt{5}+4\sqrt{2}}{9}.$$

(6) If $\cos\alpha = \frac{3}{4}$, $\sin\alpha = \frac{\sqrt{7}}{4}$.

If $\cos\beta = \frac{2}{5}$, $\sin\beta = \frac{\sqrt{21}}{5}$.

$$\therefore\ \sin(\alpha-\beta) = \frac{\sqrt{7}}{4}\cdot\frac{2}{5} - \frac{3}{4}\cdot\frac{\sqrt{21}}{5} = \frac{2\sqrt{7}-3\sqrt{21}}{20}.$$

(7) If $\sin\alpha = \cdot 5 = \frac{1}{2}$, $\cos\alpha = \frac{\sqrt{3}}{2}$.

If $\cos\beta = \frac{1}{\sqrt{2}}$, $\sin\beta = \frac{1}{\sqrt{2}}$;

$$\therefore\ \cos(\alpha+\beta) = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

(8) If $\cos\alpha = \cdot 0\dot{3} = \frac{1}{30}$, $\sin\alpha = \frac{\sqrt{899}}{30}$.

If $\sin\beta = \frac{1}{2}$, $\cos\beta = \frac{\sqrt{3}}{2}$;

$$\therefore\ \cos(\alpha-\beta) = \frac{1}{30}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{899}}{30}\cdot\frac{1}{2} = \frac{\sqrt{3}+\sqrt{899}}{60}.$$

Examples—XXIX. (p. 88).

(1)
$$\begin{aligned}\cos(90^\circ + A) &= \cos 90^\circ.\ \cos A - \sin 90^\circ.\ \sin A\\ &= 0\,.\ \cos A - 1\,.\ \sin A = -\sin A.\end{aligned}$$

(2)
$$\begin{aligned}\sin(180^\circ + A) &= \sin 180^\circ.\ \cos A + \cos 180^\circ.\ \sin A\\ &= 0\,.\ \cos A - 1\,.\ \sin A = -\sin A.\end{aligned}$$

(3)
$$\begin{aligned}\cos(\pi+\theta) &= \cos\pi\,.\ \cos\theta - \sin\pi\,.\ \sin\theta\\ &= -1\,.\ \cos\theta - 0\,.\ \sin\theta = -\cos\theta.\end{aligned}$$

(4)
$$\begin{aligned}\sin\left(\frac{3\pi}{2}+\theta\right) &= \sin\frac{3\pi}{2}\cdot\cos\theta + \cos\frac{3\pi}{2}\cdot\sin\theta\\ &= -1\,.\ \cos\theta + 0\,.\ \sin\theta = -\cos\theta.\end{aligned}$$

(5) $$\operatorname{cosec}\left(\frac{\pi}{2}+a\right)=\frac{1}{\sin\left(\frac{\pi}{2}+a\right)}$$

$$=\frac{1}{\sin\frac{\pi}{2}\cdot\cos a+\cos\frac{\pi}{2}\cdot\sin a}$$

$$=\frac{1}{1\,.\,\cos a+0\,.\,\sin a}=\frac{1}{\cos a}=\sec a.$$

(6) $$\tan(\pi+a)=\frac{\sin(\pi+a)}{\cos(\pi+a)}=\frac{0\,.\,\cos a-1\,.\,\sin a}{-1\,.\,\cos a-0\,.\,\sin a}=\frac{-\sin a}{-\cos a}=\tan a.$$

(7) $$\sin(2\pi-\theta)=\sin 2\pi\,.\,\cos\theta-\cos 2\pi\,.\,\sin\theta$$
$$=0\,.\,\cos\theta-1\,.\,\sin\theta=-\sin\theta.$$

(8) $$\tan(2\pi-\theta)=\frac{\sin(2\pi-\theta)}{\cos(2\pi-\theta)}=\frac{0\,.\,\cos\theta-1\,.\,\sin\theta}{1\,.\,\cos\theta+0\,.\,\sin\theta}=\frac{-\sin\theta}{\cos\theta}=-\tan\theta.$$

(9) $$\sec(180^\circ-\theta)=\frac{1}{\cos(180^\circ-\theta)}=\frac{1}{-1\,.\,\cos\theta+0\,.\,\sin\theta}=-\frac{1}{\cos\theta}=-\sec\theta.$$

(10) $$\operatorname{cosec}(\pi-\theta)=\frac{1}{\sin(\pi-\theta)}=\frac{1}{0\,.\,\cos\theta-(-1\,.\,\sin\theta)}=\frac{1}{\sin\theta}=\operatorname{cosec}\theta.$$

EXAMPLES—XXX. (p. 89).

(1) $$\sin\theta-\cos\theta=0.$$
$$\sin\theta\cdot\frac{1}{\sqrt{2}}-\cos\theta\cdot\frac{1}{\sqrt{2}}=0$$
$$\sin\theta\,.\,\cos 45^\circ-\cos\theta\,.\,\sin 45^\circ=0;$$
$$\therefore\ \sin(\theta-45^\circ)=0,\ \therefore\ \theta-45^\circ=0^\circ,\text{ or }\theta=45^\circ.$$

(2) $$\sin\theta+\cos\theta=1$$
$$\sin\theta\cdot\frac{1}{\sqrt{2}}+\cos\theta\cdot\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}};$$
$$\sin\theta\,.\,\cos 45^\circ+\cos\theta\,.\,\sin 45^\circ=\frac{1}{\sqrt{2}};$$
$$\therefore\ \sin(\theta+45^\circ)=\sin 45^\circ;$$
$$\therefore\ \theta+45^\circ=45^\circ,\text{ or, }\theta=0^\circ.$$

(3) $$\sin\theta - \cos\theta = \sqrt{\frac{3}{2}}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} - \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore \sin(\theta - 45^\circ) = \sin 60^\circ;$$

$$\therefore \theta - 45^\circ = 60^\circ, \text{ or, } \theta = 105^\circ.$$

(4) $$\sin\theta + \cos\theta = \frac{\sqrt{3}+1}{2}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\sin(\theta + 45^\circ) = \sin 75^\circ$, whence $\theta = 30^\circ$, or,
$\cos(\theta - 45^\circ) = \cos 15^\circ$, whence $\theta = 60^\circ$, or,
$\cos(45^\circ - \theta) = \cos 15^\circ$, whence $\theta = -30^\circ$.

(5) $$\sin\theta + \cos\theta = \sqrt{2}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{\sqrt{2}} = 1$$

$\sin(\theta + 45^\circ) = \sin 90^\circ$, or, $\theta = 45^\circ$.

(6) $$\sin\theta - \cos\theta = \frac{\sqrt{3}-1}{2}$$

$$\sin\theta \cdot \frac{1}{\sqrt{2}} - \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$\sin(\theta - 45^\circ) = \sin 15^\circ$, whence $\theta = 60^\circ$.

EXAMPLES—XXXI. (p. 92).

(1) $\sin 6A + \sin 4A = 2\sin\frac{6A+4A}{2} \cdot \cos\frac{6A-4A}{2} = 2\sin 5A \cdot \cos A.$

(2) $\sin 5A - \sin 3A = 2\cos\frac{5A+3A}{2} \cdot \sin\frac{5A-3A}{2} = 2\cos 4A \cdot \sin A.$

(3) $\cos 7\theta + \cos 9\theta = 2\cos\frac{7\theta+9\theta}{2} \cdot \cos\frac{9\theta-7\theta}{2} = 2\cos 8\theta \cdot \cos\theta.$

(4) $\cos\theta - \cos 5\theta = 2\sin\frac{\theta+5\theta}{2} \cdot \sin\frac{5\theta-\theta}{2} = 2\sin 3\theta \cdot \sin 2\theta.$

(5) $\sin\alpha + \sin4\alpha = 2\sin\dfrac{\alpha+4\alpha}{2}\cdot\cos\dfrac{4\alpha-\alpha}{2} = 2\sin\dfrac{5\alpha}{2}\cdot\cos\dfrac{3\alpha}{2}.$

(6) $\cos5\alpha - \cos8\alpha = 2\sin\dfrac{5\alpha+8\alpha}{2}\cdot\sin\dfrac{8\alpha-5\alpha}{2} = 2\sin\dfrac{13\alpha}{2}\cdot\sin\dfrac{3\alpha}{2}.$

(7) $2\sin5\theta\,.\,\cos7\theta = \sin(5\theta+7\theta) - \sin(7\theta-5\theta) = \sin12\theta - \sin2\theta.$

(8) $2\sin3\theta\,.\,\sin5\theta = \cos(5\theta-3\theta) - \cos(5\theta+3\theta) = \cos2\theta - \cos8\theta.$

(9) $2\cos\alpha\,.\,\cos4\alpha = \cos(\alpha+4\alpha) + \cos(4\alpha-\alpha) = \cos5\alpha + \cos3\alpha.$

(10) $2\cos\alpha\,.\,\sin2\alpha = \sin(\alpha+2\alpha) + \sin(2\alpha-\alpha) = \sin3\alpha + \sin\alpha.$

(11) $$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\dfrac{A+B}{2}\cdot\cos\dfrac{A-B}{2}}{2\cos\dfrac{A+B}{2}\cdot\cos\dfrac{A-B}{2}} = \frac{\sin\dfrac{A+B}{2}}{\cos\dfrac{A+B}{2}} = \tan\frac{A+B}{2}.$$

(12) $$\frac{\cos A - \cos3A}{\sin3A - \sin A} = \frac{2\sin2A\,.\,\sin A}{2\cos2A\,.\,\sin A} = \frac{\sin2A}{\cos2A} = \tan2A.$$

(13) $$\frac{\sin2A + \sin A}{\cos2A + \cos A} = \frac{2\sin\dfrac{3A}{2}\cdot\cos\dfrac{A}{2}}{2\cos\dfrac{3A}{2}\cdot\cos\dfrac{A}{2}} = \frac{\sin\dfrac{3A}{2}}{\cos\dfrac{3A}{2}} = \tan\frac{3A}{2}.$$

(14) $\cos(30^\circ-\theta) - \cos(30^\circ+\theta) = 2\sin30^\circ\cdot\sin\theta = 2\times\dfrac{1}{2}\cdot\sin\theta = \sin\theta.$

(15) $\cos\left(\dfrac{\pi}{3}+\theta\right) + \cos\left(\dfrac{\pi}{3}-\theta\right) = 2\cos\dfrac{\pi}{3}\cdot\cos\theta = 2\times\dfrac{1}{2}\cdot\cos\theta = \cos\theta.$

(16) $\sin\left(\dfrac{\pi}{3}+\alpha\right) - \sin\left(\dfrac{\pi}{3}-\alpha\right) = 2\cos\dfrac{\pi}{3}\cdot\sin\alpha = 2\times\dfrac{1}{2}\cdot\sin\alpha = \sin\alpha.$

(17) $$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \frac{2\cos\dfrac{\alpha+\beta}{2}\cdot\sin\dfrac{\alpha-\beta}{2}}{2\sin\dfrac{\alpha+\beta}{2}\cdot\sin\dfrac{\alpha-\beta}{2}} = \frac{\cos\dfrac{\alpha+\beta}{2}}{\sin\dfrac{\alpha+\beta}{2}} = \cot\frac{\alpha+\beta}{2}.$$

(18) $\dfrac{\sin\alpha - \sin\beta}{\cos\beta + \cos\alpha} = \dfrac{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}} = \dfrac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = \tan\dfrac{\alpha-\beta}{2}.$

(19) $\dfrac{\sin5\theta + \sin3\theta}{\cos3\theta - \cos5\theta} = \dfrac{2\sin4\theta\,.\,\cos\theta}{2\sin4\theta\,.\,\sin\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta.$

(20) $\dfrac{\cos\alpha + \cos\beta}{\cos\beta - \cos\alpha} = \dfrac{2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}} = \dfrac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}} \div \dfrac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = \dfrac{\cot\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.$

EXAMPLES—XXXII. (p. 93).

(1)

$\sin\alpha - \cos\beta = \sin\alpha - \sin\left(\frac{\pi}{2} - \beta\right) = 2\cos\frac{1}{2}\left(\alpha + \frac{\pi}{2} - \beta\right).\sin\frac{1}{2}\left(\alpha - \frac{\pi}{2} + \beta\right).$

(2)

$\sin\left(\frac{\pi}{2} + \alpha\right) + \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\left(\frac{\pi}{2} + \alpha\right) + \sin\alpha = 2\sin\left(\frac{\pi}{4} + \alpha\right)\cdot\cos\frac{\pi}{4}.$

(3) $\sin\alpha + \cos\alpha = \sin\alpha + \sin\left(\frac{\pi}{2} - \alpha\right) = 2\sin\frac{\pi}{4}\cdot\cos\left(\alpha - \frac{\pi}{4}\right).$

(4) $\sin\alpha - \cos\alpha = \sin\alpha - \sin\left(\frac{\pi}{2} - \alpha\right) = 2\cos\frac{\pi}{4}\cdot\sin\left(\alpha - \frac{\pi}{4}\right).$

(5) $\sin30° + \cos80° = \sin30° + \sin10° = 2\sin20°.\cos10°.$

(6) $\sin20° - \cos80° = \sin20° - \sin10° = 2\cos15°.\sin5°.$

(7) $\sin\frac{\pi}{4} + \cos\frac{\pi}{6} = \sin\frac{\pi}{4} + \sin\frac{\pi}{3} = 2\sin\frac{7\pi}{24}\cdot\cos\frac{\pi}{24}.$

(8) $\sin\frac{\pi}{3} - \cos\frac{\pi}{5} = \sin\frac{\pi}{3} - \sin\frac{3\pi}{10} = 2\cos\frac{19\pi}{60}\cdot\sin\frac{\pi}{60}.$

EXAMPLES—XXXIII. (p. 96).

(1) $$\frac{\tan a+\tan\beta}{\cot a+\cot\beta}=\frac{\dfrac{\sin a}{\cos a}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\cos a}{\sin a}+\dfrac{\cos\beta}{\sin\beta}}=\frac{\dfrac{\sin a\,.\cos\beta+\cos a\,.\sin\beta}{\cos a\,.\cos\beta}}{\dfrac{\cos a\,.\sin\beta+\sin a\,.\cos\beta}{\sin a\,.\sin\beta}}.$$

$$=\frac{\sin a\,.\sin\beta}{\cos a\,.\cos\beta}=\tan a\,.\tan\beta.$$

(2) $$\frac{\tan a+\tan\beta}{\cot a-\tan\beta}=\frac{\dfrac{\sin a}{\cos a}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\cos a}{\sin a}-\dfrac{\sin\beta}{\cos\beta}}=\frac{\dfrac{\sin a\,.\cos\beta+\cos a\,.\sin\beta}{\cos a\,.\cos\beta}}{\dfrac{\cos a\,.\cos\beta-\sin a\,.\sin\beta}{\sin a\,.\cos\beta}}.$$

$$=\frac{\sin(a+\beta)\,.\sin a\,.\cos\beta}{\cos(a+\beta)\,.\cos a\,.\cos\beta}=\tan(a+\beta)\,.\tan a.$$

(3) $$\frac{\tan a-\tan\beta}{\cot a+\tan\beta}=\frac{\dfrac{\sin a}{\cos a}-\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\cos a}{\sin a}+\dfrac{\sin\beta}{\cos\beta}}=\frac{\dfrac{\sin a\,.\cos\beta-\cos a\,.\sin\beta}{\cos a\,.\cos\beta}}{\dfrac{\cos a\,.\cos\beta+\sin a\,.\sin\beta}{\sin a\,.\cos\beta}}.$$

$$=\frac{\sin(a-\beta)\,.\sin a\,.\cos\beta}{\cos(a-\beta)\,.\cos a\,.\cos\beta}=\tan(a-\beta)\,.\tan a.$$

(4) $$\tan\frac{\phi+\psi}{2}+\tan\frac{\phi-\psi}{2}=\frac{\sin\dfrac{\phi+\psi}{2}}{\cos\dfrac{\phi+\psi}{2}}+\frac{\sin\dfrac{\phi-\psi}{2}}{\cos\dfrac{\phi-\psi}{2}}$$

$$=\frac{\sin\dfrac{\phi+\psi}{2}\cdot\cos\dfrac{\phi-\psi}{2}+\cos\dfrac{\phi+\psi}{2}\cdot\sin\dfrac{\phi-\psi}{2}}{\cos\dfrac{\phi+\psi}{2}\cdot\cos\dfrac{\phi-\psi}{2}}$$

$$=\frac{\sin\left(\dfrac{\phi+\psi}{2}+\dfrac{\phi-\psi}{2}\right)}{\dfrac{1}{2}(\cos\phi+\cos\psi)}=\frac{\sin\phi}{\dfrac{1}{2}(\cos\phi+\cos\psi)}=\frac{2\sin\phi}{\cos\phi+\cos\psi}.$$

(5) $\sin\phi=\sin\{\psi+(\phi-\psi)\}=\sin\psi\,.\cos(\phi-\psi)+\cos\psi\,.\sin(\phi-\psi).$

(6) $\cos\phi=\cos\{(\phi+\psi)-\psi\}=\cos(\phi+\psi)\,.\cos\psi+\sin(\phi+\psi)\,.\sin\psi.$

(7) $(\cos\alpha+\cos\beta)\{1-\cos(\alpha+\beta)\}=(\cos\alpha+\cos\beta)(1-\cos\alpha.\cos\beta+\sin\alpha.\sin\beta)$

$=\cos\alpha+\cos\beta-\cos^2\alpha.\cos\beta-\cos\alpha.\cos^2\beta+\sin\alpha.\sin\beta.\cos\alpha+\sin\alpha\,\sin\beta.\cos\beta$

$=\cos\alpha(1-\cos^2\beta)+\cos\beta(1-\cos^2\alpha)+\sin\alpha.\sin\beta.\cos\alpha+\sin\alpha.\sin\beta.\cos\beta$

$=\cos\alpha.\sin^2\beta+\cos\beta.\sin^2\alpha+\sin\alpha.\sin\beta.\cos\alpha+\sin\alpha.\sin\beta.\cos\beta$

$=\sin\beta(\cos\alpha.\sin\beta)+\sin\alpha(\cos\beta.\sin\alpha)+\sin\alpha(\sin\beta.\cos\alpha)+\sin\beta.(\sin\alpha.\cos\beta)$

$=\sin\beta.(\cos\alpha.\sin\beta+\sin\alpha.\cos\beta)+\sin\alpha(\cos\beta.\sin\alpha.+\sin\beta.\cos\alpha)$

$=\sin\beta.\sin(\alpha+\beta)+\sin\alpha.\sin(\alpha+\beta)$

$=(\sin\alpha+\sin\beta).\sin(\alpha+\beta).$

(8)

$$\frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta}=\frac{\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha+\beta}{2}\right)}{\sin\alpha+\sin\beta}=\frac{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}$$

$$=\frac{2\cos\frac{\alpha+\beta}{2}}{2\cos\frac{\alpha-\beta}{2}}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

(9)

$$\frac{\sin(\alpha+\beta)}{\sin\alpha-\sin\beta}=\frac{\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha+\beta}{2}\right)}{\sin\alpha-\sin\beta}=\frac{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}$$

$$=\frac{2\sin\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha-\beta}{2}}=\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}.$$

(10) $$\cot\frac{\alpha+\beta}{2}+\cot\frac{\alpha-\beta}{2}=\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}}+\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\alpha-\beta}{2}}$$

$$=\frac{\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}+\cos\frac{\alpha-\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)}{\frac{1}{2}(\cos\beta-\cos\alpha)}=\frac{\sin\alpha}{\frac{1}{2}(\cos\beta-\cos\alpha)}=\frac{2\sin\alpha}{\cos\beta-\cos\alpha}.$$

(11) $$\tan\frac{\alpha+\beta}{2}-\tan\frac{\alpha-\beta}{2}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}}-\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$=\frac{\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)}{\frac{1}{2}(\cos\alpha+\cos\beta)}=\frac{\sin\beta}{\frac{1}{2}(\cos\alpha+\cos\beta)}=\frac{2\sin\beta}{\cos\alpha+\cos\beta}.$$

(12) $$\frac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta}=\frac{2\sin\frac{\beta+\alpha}{2}\cdot\sin\frac{\beta-\alpha}{2}}{2\sin\frac{\beta+\alpha}{2}\cdot\cos\frac{\beta-\alpha}{2}}=\tan\frac{\beta-\alpha}{2}.$$

(13) $$\cot\beta-\tan\alpha=\frac{\cos\beta}{\sin\beta}-\frac{\sin\alpha}{\cos\alpha}=\frac{\cos\alpha\,.\cos\beta-\sin\alpha\,.\sin\beta}{\cos\alpha\,.\sin\beta}=\frac{\cos(\alpha+\beta)}{\cos\alpha\,.\sin\beta}.$$

(14) $$\cot\theta+\tan\phi=\frac{\cos\theta}{\sin\theta}+\frac{\sin\phi}{\cos\phi}=\frac{\cos\theta\,.\cos\phi+\sin\theta\,.\sin\phi}{\sin\theta\,.\cos\phi}=\frac{\cos(\phi-\theta)}{\sin\theta\,.\cos\phi}.$$

(15) $$\tan^2\alpha - \tan^2\beta = \frac{\sin^2\alpha}{\cos^2\alpha} - \frac{\sin^2\beta}{\cos^2\beta} = \frac{\sin^2\alpha . \cos^2\beta - \cos^2\alpha . \sin^2\beta}{\cos^2\alpha . \cos^2\beta}$$
$$= \frac{(\sin\alpha . \cos\beta + \cos\alpha . \sin\beta)(\sin\alpha . \cos\beta - \cos\alpha . \sin\beta)}{\cos^2\alpha . \cos^2\beta}$$
$$= \frac{\sin(\alpha+\beta) . \sin(\alpha-\beta)}{\cos^2\alpha . \cos^2\beta}.$$

(16) $$1 + \tan\alpha.\tan\beta = 1 + \frac{\sin\alpha.\sin\beta}{\cos\alpha.\cos\beta} = \frac{\cos\alpha.\cos\beta + \sin\alpha.\sin\beta}{\cos\alpha . \cos\beta} = \frac{\cos(\alpha-\beta)}{\cos\alpha.\cos\beta}.$$

(17) $$1 - \tan\alpha . \tan\beta = 1 - \frac{\sin\alpha . \sin\beta}{\cos\alpha . \cos\beta} = \frac{\cos\alpha . \cos\beta - \sin\alpha . \sin\beta}{\cos\alpha . \cos\beta}$$
$$= \frac{\cos(\alpha+\beta)}{\cos\alpha . \cos\beta}.$$

(18) $$\frac{\cot\alpha + \tan\beta}{\tan\alpha + \cot\beta} = \frac{\dfrac{\cos\alpha}{\sin\alpha} + \dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\beta}{\sin\beta}} = \frac{\dfrac{\cos\alpha . \cos\beta + \sin\alpha . \sin\beta}{\sin\alpha . \cos\beta}}{\dfrac{\sin\alpha . \sin\beta + \cos\alpha . \cos\beta}{\cos\alpha . \sin\beta}}$$
$$= \frac{\cos\alpha . \sin\beta}{\sin\alpha . \cos\beta} = \cot\alpha . \tan\beta.$$

(19) $$\frac{\tan^2x - \tan^2y}{1 - \tan^2x . \tan^2y} = \frac{\dfrac{\sin^2x}{\cos^2x} - \dfrac{\sin^2y}{\cos^2y}}{1 - \dfrac{\sin^2x . \sin^2y}{\cos^2x . \cos^2y}} = \frac{\sin^2x . \cos^2y - \cos^2x . \sin^2y}{\cos^2x . \cos^2y - \sin^2x . \sin^2y}$$
$$= \frac{(\sin x . \cos y + \cos x . \sin y)(\sin x . \cos y - \cos x . \sin y)}{(\cos x . \cos y + \sin x . \sin y)(\cos x . \cos y - \sin x . \sin y)}$$
$$= \frac{\sin(x+y) . \sin(x-y)}{\cos(x-y) . \cos(x+y)} = \tan(x+y) . \tan(x-y).$$

(20) $$\cot(\theta + 45^\circ) = \frac{\cos(\theta+45^\circ)}{\sin(\theta+45^\circ)} = \frac{\cos\theta \cdot \dfrac{1}{\sqrt{2}} - \sin\theta \cdot \dfrac{1}{\sqrt{2}}}{\sin\theta \cdot \dfrac{1}{\sqrt{2}} + \cos\theta \cdot \dfrac{1}{\sqrt{2}}} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$$
$$= \frac{\dfrac{\cos\theta}{\sin\theta} - 1}{1 + \dfrac{\cos\theta}{\sin\theta}} = \frac{\cot\theta - 1}{\cot\theta + 1}.$$

(21) $$\sin\theta+\cos\theta=\sqrt{2}.\left(\sin\theta\cdot\frac{1}{\sqrt{2}}+\cos\theta\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}.\sin(45^\circ+\theta).$$

(22) $$\cos\theta-\sin\theta=\sqrt{2}\left(\cos\theta\cdot\frac{1}{\sqrt{2}}-\sin\theta\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}.\sin\left(\frac{\pi}{4}-\theta\right).$$

(23) $$\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}=\frac{\sin\alpha.\cos\beta-\cos\alpha.\sin\beta}{\sin\alpha.\cos\beta+\cos\alpha.\sin\beta}=\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}.$$

(24) $$\frac{\cot x-\cot y}{\cot x+\cot y}=\frac{\dfrac{\cos x}{\sin x}-\dfrac{\cos y}{\sin y}}{\dfrac{\cos x}{\sin x}+\dfrac{\cos y}{\sin y}}=\frac{\cos x.\sin y-\sin x.\cos y}{\cos x.\sin y+\sin x.\cos y}=\frac{\sin(y-x)}{\sin(y+x)}.$$

(25) $$\begin{aligned}\cos(A-B)+\sin(A+B)&=\cos(A-B)+\cos(90^\circ-A-B)\\&=2\cos(45^\circ-B).\cos(45^\circ-A)\\&=2\cos(B-45^\circ).\sin(45^\circ+A).\end{aligned}$$

(26) $$\begin{aligned}\cos(A-B)-\sin(A+B)&=\sin(90^\circ-A+B)-\sin(A+B)\\&=2\cos(45^\circ+B).\sin(45^\circ-A).\end{aligned}$$

(27) $$\begin{aligned}\cos(A+B)+\sin(A-B)&=\cos(A+B)+\cos(90^\circ-A+B)\\&=2\cos(45^\circ+B).\cos(45^\circ-A)\\&=2\cos(45^\circ+B).\sin(45^\circ+A).\end{aligned}$$

(28) $$\begin{aligned}\cos(A+B)-\sin(A-B)&=\sin(90^\circ-A-B)-\sin(A-B)\\&=2\cos(45^\circ-B).\sin(45^\circ-A).\end{aligned}$$

(29) $$\begin{aligned}\frac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}&=\frac{\cos\alpha+\cos\beta}{-(\cos\beta-\cos\alpha)}\\&=-\frac{2\cos\dfrac{\alpha+\beta}{2}.\cos\dfrac{\alpha-\beta}{2}}{2\sin\dfrac{\alpha+\beta}{2}.\sin\dfrac{\alpha-\beta}{2}}=-\frac{\cot\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.\end{aligned}$$

(30) $$\begin{aligned}\sec 72^\circ-\sec 36^\circ&=\frac{1}{\cos 72^\circ}-\frac{1}{\cos 36^\circ}=\frac{\cos 36^\circ-\cos 72^\circ}{\cos 72^\circ.\cos 36^\circ}\\&=\frac{2\sin 54^\circ.\sin 18^\circ}{\sin 18^\circ.\sin 54^\circ}=2=\sec 60^\circ.\end{aligned}$$

(31) $(\sin 81^\circ + \sin 9^\circ)(\sin 81^\circ - \sin 9^\circ)$

$$= (2\sin 45^\circ . \cos 36^\circ).(2\cos 45^\circ . \sin 36^\circ)$$
$$= 2\cdot\frac{1}{\sqrt{2}}\cdot\sin 54^\circ . 2\cdot\frac{1}{\sqrt{2}}\cdot\cos 54^\circ$$
$$= 2\sin 54^\circ . \cos 54^\circ$$
$$= \sin 108^\circ.$$

(32) $$\frac{\cos 3^\circ - \cos 33^\circ}{\sin 3^\circ + \sin 33^\circ} = \frac{2\sin 18^\circ . \sin 15^\circ}{2\sin 18^\circ . \cos 15^\circ} = \tan 15^\circ.$$

(33) $$\frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ} = \frac{2\sin 18^\circ . \cos 15^\circ}{2\cos 18^\circ . \cos 15^\circ} = \tan 18^\circ.$$

(34) $$\frac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \frac{\sin 81^\circ + \sin 9^\circ}{\sin 81^\circ - \sin 9^\circ} = \frac{2\sin 45^\circ . \cos 36^\circ}{2\cos 45^\circ . \sin 36^\circ} = \cot 36^\circ = \tan 54^\circ.$$

(35) $$\frac{\cos 27^\circ - \sin 27^\circ}{\cos 27^\circ + \sin 27^\circ} = \frac{\sin 63^\circ - \sin 27^\circ}{\sin 63^\circ + \sin 27^\circ} = \frac{2\cos 45^\circ . \sin 18^\circ}{2\sin 45^\circ . \cos 18^\circ} = \tan 18^\circ.$$

(36) $\tan 50^\circ + \cot 50^\circ = \tan 50^\circ + \tan 40^\circ$

$$= \frac{\sin 50^\circ . \cos 40^\circ + \cos 50^\circ . \sin 40^\circ}{\cos 50^\circ . \cos 40^\circ} = \frac{\sin 90^\circ}{\frac{1}{2}\{\cos 90^\circ + \cos 10^\circ\}}$$
$$= \frac{2\sin 90^\circ}{\cos 10^\circ} = \frac{2}{\cos 10^\circ} = 2\sec 10^\circ.$$

EXAMPLES—XXXIV. (p. 100).

(1) $$\frac{2\cot A}{1+\cot^2 A} = \frac{2\cot A}{\operatorname{cosec}^2 A} = \frac{2\cos A}{\sin A}\cdot\sin^2 A = 2\cos A . \sin A = \sin 2A.$$

(2) $$\frac{\sin 2A}{1+\cos 2A}\cdot\frac{\cos A}{1+\cos A} = \frac{2\sin A . \cos A}{2\cos^2 A}\cdot\frac{\cos A}{1+\cos A} = \frac{\sin A}{1+\cos A}$$
$$= \frac{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{2\cos^2\frac{A}{2}} = \tan\frac{A}{2}.$$

(3) $$\operatorname{cosec} A + \cot A = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \frac{1+\cos A}{\sin A} = \frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}} = \cot\frac{A}{2}.$$

(21) $\sin\theta+\cos\theta=\sqrt{2}.\left(\sin\theta\cdot\frac{1}{\sqrt{2}}+\cos\theta\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}\,.\sin(45^\circ+\theta).$

(22) $\cos\theta-\sin\theta=\sqrt{2}\left(\cos\theta\cdot\frac{1}{\sqrt{2}}-\sin\theta\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}.\sin\left(\frac{\pi}{4}-\theta\right).$

(23) $$\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}=\frac{\sin\alpha\,.\cos\beta-\cos\alpha\,.\sin\beta}{\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta}=\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}.$$

(24) $$\frac{\cot x-\cot y}{\cot x+\cot y}=\frac{\frac{\cos x}{\sin x}-\frac{\cos y}{\sin y}}{\frac{\cos x}{\sin x}+\frac{\cos y}{\sin y}}=\frac{\cos x\,.\sin y-\sin x\,.\cos y}{\cos x\,.\sin y+\sin x\,.\cos y}=\frac{\sin(y-x)}{\sin(y+x)}.$$

(25) $$\begin{aligned}\cos(A-B)+\sin(A+B)&=\cos(A-B)+\cos(90^\circ-A-B)\\&=2\cos(45^\circ-B)\,.\cos(45^\circ-A)\\&=2\cos(B-45^\circ)\,.\sin(45^\circ+A).\end{aligned}$$

(26) $$\begin{aligned}\cos(A-B)-\sin(A+B)&=\sin(90^\circ-A+B)-\sin(A+B)\\&=2\cos(45^\circ+B)\,.\sin(45^\circ-A).\end{aligned}$$

(27) $$\begin{aligned}\cos(A+B)+\sin(A-B)&=\cos(A+B)+\cos(90^\circ-A+B)\\&=2\cos(45^\circ+B)\,.\cos(45^\circ-A)\\&=2\cos(45^\circ+B)\,.\sin(45^\circ+A).\end{aligned}$$

(28) $$\begin{aligned}\cos(A+B)-\sin(A-B)&=\sin(90^\circ-A-B)-\sin(A-B)\\&=2\cos(45^\circ-B)\,.\sin(45^\circ-A).\end{aligned}$$

(29) $$\begin{aligned}\frac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}&=\frac{\cos\alpha+\cos\beta}{-(\cos\beta-\cos\alpha)}\\&=-\frac{2\cos\frac{\alpha+\beta}{2}\,.\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\,.\sin\frac{\alpha-\beta}{2}}=-\frac{\cot\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.\end{aligned}$$

(30) $$\begin{aligned}\sec72^\circ-\sec36^\circ&=\frac{1}{\cos72^\circ}-\frac{1}{\cos36^\circ}=\frac{\cos36^\circ-\cos72^\circ}{\cos72^\circ.\cos36^\circ}\\&=\frac{2\sin54^\circ.\sin18^\circ}{\sin18^\circ.\sin54^\circ}=2=\sec60^\circ.\end{aligned}$$

(31) $$(\sin 81^\circ+\sin 9^\circ)(\sin 81^\circ-\sin 9^\circ)$$
$$=(2\sin 45^\circ.\cos 36^\circ).(2\cos 45^\circ.\sin 36^\circ)$$
$$=2\cdot\frac{1}{\sqrt{2}}\cdot\sin 54^\circ.\,2\cdot\frac{1}{\sqrt{2}}\cdot\cos 54^\circ$$
$$=2\sin 54^\circ.\cos 54^\circ$$
$$=\sin 108^\circ.$$

(32) $$\frac{\cos 3^\circ-\cos 33^\circ}{\sin 3^\circ+\sin 33^\circ}=\frac{2\sin 18^\circ.\sin 15^\circ}{2\sin 18^\circ.\cos 15^\circ}=\tan 15^\circ.$$

(33) $$\frac{\sin 33^\circ+\sin 3^\circ}{\cos 33^\circ+\cos 3^\circ}=\frac{2\sin 18^\circ.\cos 15^\circ}{2\cos 18^\circ.\cos 15^\circ}=\tan 18^\circ.$$

(34) $$\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\frac{\sin 81^\circ+\sin 9^\circ}{\sin 81^\circ-\sin 9^\circ}=\frac{2\sin 45^\circ.\cos 36^\circ}{2\cos 45^\circ.\sin 36^\circ}=\cot 36^\circ=\tan 54^\circ.$$

(35) $$\frac{\cos 27^\circ-\sin 27^\circ}{\cos 27^\circ+\sin 27^\circ}=\frac{\sin 63^\circ-\sin 27^\circ}{\sin 63^\circ+\sin 27^\circ}=\frac{2\cos 45^\circ.\sin 18^\circ}{2\sin 45^\circ.\cos 18^\circ}=\tan 18^\circ.$$

(36) $$\tan 50^\circ+\cot 50^\circ=\tan 50^\circ+\tan 40^\circ$$
$$=\frac{\sin 50^\circ.\cos 40^\circ+\cos 50^\circ.\sin 40^\circ}{\cos 50^\circ.\cos 40^\circ}=\frac{\sin 90^\circ}{\frac{1}{2}\{\cos 90^\circ+\cos 10^\circ\}}$$
$$=\frac{2\sin 90^\circ}{\cos 10^\circ}=\frac{2}{\cos 10^\circ}=2\sec 10^\circ.$$

EXAMPLES—XXXIV. (p. 100).

(1) $$\frac{2\cot A}{1+\cot^2 A}=\frac{2\cot A}{\operatorname{cosec}^2 A}=\frac{2\cos A}{\sin A}\cdot\sin^2 A=2\cos A.\sin A=\sin 2A.$$

(2) $$\frac{\sin 2A}{1+\cos 2A}\cdot\frac{\cos A}{1+\cos A}=\frac{2\sin A.\cos A}{2\cos^2 A}\cdot\frac{\cos A}{1+\cos A}=\frac{\sin A}{1+\cos A}$$
$$=\frac{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{2\cos^2\frac{A}{2}}=\tan\frac{A}{2}.$$

(3) $$\operatorname{cosec} A+\cot A=\frac{1}{\sin A}+\frac{\cos A}{\sin A}=\frac{1+\cos A}{\sin A}=\frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\cot\frac{A}{2}.$$

(4)

$$\tan\theta+\cot\theta=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\,.\cos\theta}=\frac{1}{\sin\theta.\cos\theta}=\frac{2}{2\sin\theta.\cos\theta}=\frac{2}{\sin2\theta}.$$

(5) $$\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2\,\frac{\sin\theta}{\cos\theta}}{\sec^2\theta}=\frac{2\sin\theta}{\cos\theta}\cdot\cos^2\theta=2\sin\theta\,.\cos\theta=\sin2\theta.$$

(6) $$2\operatorname{cosec}2A=\frac{2}{\sin2A}=\frac{2}{2\sin A.\cos A}=\operatorname{cosec}A\,.\sec A.$$

(7) $$\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}=\cos^2\theta-\sin^2\theta=\cos2\theta.$$

(8) $$\frac{2\sec2\theta}{1+\sec2\theta}=\frac{\frac{2}{\cos2\theta}}{1+\frac{1}{\cos2\theta}}=\frac{2}{\cos2\theta+1}=\frac{2}{2\cos^2\theta}=\sec^2\theta.$$

(9) $$\frac{1-\tan A}{1+\tan A}=\frac{\cos A-\sin A}{\cos A+\sin A}=\frac{\cos^2A-\sin^2A}{(\cos A+\sin A)^2}=\frac{1-2\sin^2A}{1+\sin2A}.$$

(10) $$\cot\theta-2\cot2\theta=\frac{\cos\theta}{\sin\theta}-\frac{2\cos2\theta}{\sin2\theta}=\frac{\cos\theta}{\sin\theta}-\frac{\cos2\theta}{\sin\theta\,.\cos\theta}.$$

$$=\frac{\cos^2\theta-\cos2\theta}{\sin\theta\,.\cos\theta}=\frac{\cos^2\theta-2\cos^2\theta+1}{\sin\theta\,.\cos\theta}=\frac{\sin^2\theta}{\sin\theta\,.\cos\theta}=\tan\theta.$$

(11) $$\frac{1-\cos a}{\sin a}=\frac{2\sin^2\frac{a}{2}}{2\sin\frac{a}{2}\cdot\cos\frac{a}{2}}=\frac{\sin\frac{a}{2}}{\cos\frac{a}{2}}=\tan\frac{a}{2}\cdot$$

(12) $$\frac{2\sqrt{(\operatorname{cosec}^2\phi-1)}}{\operatorname{cosec}^2\phi}=\frac{2\,.\cot\phi}{\operatorname{cosec}^2\phi}=\frac{2.\cos\phi.\sin^2\phi}{\sin\phi}=2\sin\phi\,.\cos\phi=\sin2\phi.$$

(13) $$\frac{2-\sec^2\phi}{\sec^2\phi}=2\cos^2\phi-1=\cos2\phi.$$

(14) $$\frac{2\cot\phi}{\cot^2\phi-1}=\frac{2\cos\phi\,.\,\sin\phi}{\cos^2\phi-\sin^2\phi}=\frac{\sin2\phi}{\cos2\phi}=\tan2\phi.$$

(15) $$\sqrt{\left(\frac{\sec2a-1}{2\sec2a}\right)}=\sqrt{\left(\frac{1-\cos2a}{2}\right)}=\sqrt{\left(\frac{1-1+2\sin^2a}{2}\right)}=\sin a.$$

(16) $$\sqrt{\left(\frac{\sec2a+1}{2\sec2a}\right)}=\sqrt{\left(\frac{1+\cos2a}{2}\right)}=\sqrt{\left(\frac{1+2\cos^2a-1}{2}\right)}=\cos a.$$

(17) $$\operatorname{cosec}2a-\cot2a=\frac{1-\cos2a}{\sin2a}=\frac{2\sin^2a}{2\sin a\,.\,\cos a}=\tan a.$$

(18) $$\operatorname{cosec}2\beta+\cot2\beta=\frac{1+\cos2\beta}{\sin2\beta}=\frac{2\cos^2\beta}{2\sin\beta\,.\,\cos\beta}=\cot\beta.$$

(19) $$\tan(45^\circ+A)=\frac{\tan45^\circ+\tan A}{1-\tan45^\circ.\,\tan A}=\frac{1+\tan A}{1-\tan A}=\frac{\cos A+\sin A}{\cos A-\sin A}$$
$$=\frac{\cos^2A-\sin^2A}{(\cos A-\sin A)^2}=\frac{\cos2A}{1-\sin2A}.$$

(20) $$\cot(45^\circ-A)=\frac{1}{\tan(45^\circ-A)}=\frac{1}{\dfrac{\tan45^\circ-\tan A}{1+\tan45^\circ.\,\tan A}}=\frac{1+\tan A}{1-\tan A}$$
$$=\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{\cos^2A-\sin^2A}=\frac{1+\sin2A}{\cos2A}=\sec2A+\tan2A.$$

(21) $$\frac{1+\sin a}{1+\cos a}=\frac{\cos^2\frac{a}{2}+\sin^2\frac{a}{2}+2\sin\frac{a}{2}\cdot\cos\frac{a}{2}}{2\cos^2\frac{a}{2}}=\frac{1}{2}+\frac{1}{2}\tan^2\frac{a}{2}+\tan\frac{a}{2}$$
$$=\frac{1}{2}\left(1+\tan^2\frac{a}{2}+2\tan\frac{a}{2}\right)=\frac{1}{2}\left(1+\tan\frac{a}{2}\right)^2.$$

(22) $$\frac{1-\sin a}{1-\cos a}=\frac{\cos^2\frac{a}{2}+\sin^2\frac{a}{2}-2\sin\frac{a}{2}\cdot\cos\frac{a}{2}}{2\sin^2\frac{a}{2}}=\frac{1}{2}\cot^2\frac{a}{2}+\frac{1}{2}-\cot\frac{a}{2}$$
$$=\frac{1}{2}\left(\cot^2\frac{a}{2}+1-2\cot\frac{a}{2}\right)=\frac{1}{2}\left(\cot\frac{a}{2}-1\right)^2.$$

(23) $$\tan\frac{\theta}{2}+\frac{1}{2}\tan\theta\,.\,\sec^2\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}+\frac{\sin\theta}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}$$

$$=\frac{2\cos\theta\,.\,\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}+\sin\theta}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}=\frac{\cos\theta\,.\,\sin\theta+\sin\theta}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}$$

$$=\frac{\sin\theta(\cos\theta+1)}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}=\frac{\sin\theta\,.\,2\cos^2\frac{\theta}{2}}{2\cos\theta\,.\,\cos^2\frac{\theta}{2}}=\tan\theta.$$

(24) $$\frac{1+\sin\theta}{1-\sin\theta}=\frac{(1+\sin\theta)^2}{1-\sin^2\theta}=\left(\frac{1+\sin\theta}{\cos\theta}\right)^2=(\sec\theta+\tan\theta)^2.$$

(25) $$\frac{1-\tan^2(45^\circ-A)}{1+\tan^2(45^\circ-A)}=\frac{\cos^2(45^\circ-A)-\sin^2(45^\circ-A)}{\cos^2(45^\circ-A)+\sin^2(45^\circ-A)}$$

$$=\frac{\cos2(45^\circ-A)}{1}=\cos(90^\circ-2A)=\sin2A.$$

(26) $$\frac{\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)}{\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)}=\frac{\frac{1+\tan\theta}{1-\tan\theta}-\frac{1-\tan\theta}{1+\tan\theta}}{\frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}}$$

$$=\frac{1+2\tan\theta+\tan^2\theta-1+2\tan\theta-\tan^2\theta}{1+2\tan\theta+\tan^2\theta+1-2\tan\theta+\tan^2\theta}$$

$$=\frac{4\tan\theta}{2+2\tan^2\theta}=\frac{2\tan\theta}{1+\tan^2\theta}=2\sin\theta\,.\,\cos\theta=\sin2\theta.$$

EXAMPLES—XXXV. (p. 103).

1. (1) $$\frac{\cos3\theta-\sin3\theta}{\sin\theta+\cos\theta}=\frac{4\cos^3\theta-3\cos\theta-3\sin\theta+4\sin^3\theta}{\sin\theta+\cos\theta}$$

$$=\frac{4(\sin^3\theta+\cos^3\theta)-3(\sin\theta+\cos\theta)}{\sin\theta+\cos\theta}$$

$$=4(\sin^2\theta-\sin\theta\,.\,\cos\theta+\cos^2\theta)-3$$

$$=1-4\sin\theta\,.\,\cos\theta=1-2\sin2\theta.$$

(2) $$\frac{2\tan\theta+\sec\theta}{1+\tan^2\theta}=\frac{2\tan\theta+\sec\theta}{\sec^2\theta}=2\tan\theta.\cos^2\theta+\cos\theta=\sin2\theta+\cos\theta.$$

(3) $$\tan\frac{A}{2}+2\sin^2\frac{A}{2}\cot A=\sin\frac{A}{2}\left\{\frac{1}{\cos\frac{A}{2}}+2\sin\frac{A}{2}\cdot\frac{\cos A}{\sin A}\right\}$$

$$=\sin\frac{A}{2}\left\{\frac{1}{\cos\frac{A}{2}}+\frac{\cos A}{\cos\frac{A}{2}}\right\}=\sin\frac{A}{2}\left(\frac{2\cos^2\frac{A}{2}}{\cos\frac{A}{2}}\right)=2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=\sin A.$$

(4) $$\frac{\cot A}{\cot A-\cot3A}+\frac{\tan A}{\tan A-\tan3A}$$

$$=\frac{\dfrac{1}{\tan A}}{\dfrac{1}{\tan A}-\dfrac{1-3\tan^2A}{\tan A(3-\tan^2A)}}+\frac{\tan A}{\tan A-\dfrac{\tan A(3-\tan^2A)}{1-3\tan^2A}}$$

$$=\frac{1}{\dfrac{3-\tan^2A-1+3\tan^2A}{3-\tan^2A}}+\frac{1}{\dfrac{1-3\tan^2A-3+\tan^2A}{1-3\tan^2A}}$$

$$=\frac{3-\tan^2A}{2(1+\tan^2A)}+\frac{1-3\tan^2A}{-2(1+\tan^2A)}$$

$$=\frac{3-\tan^2A-1+3\tan^2A}{2(1+\tan^2A)}=\frac{2+2\tan^2A}{2(1+\tan^2A)}=1.$$

(5) $$\cos4A+\cos4B=2\cos2(A+B).\cos2(A-B)$$
$$=2.\{1-2\sin^2(A+B)\}.\{1-2\sin^2(A-B)\}.$$

(6) $$\tan(45^\circ+\theta)-\tan(45^\circ-\theta)=\frac{1+\tan\theta}{1-\tan\theta}-\frac{1-\tan\theta}{1+\tan\theta}=\frac{4\tan\theta}{1-\tan^2\theta}$$

$$=\frac{\dfrac{4\sin\theta}{\cos\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}=\frac{4\sin\theta.\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{2\sin2\theta}{\cos2\theta}=\frac{2\sin^22\theta}{\cos2\theta.\sin2\theta}$$

$$=\frac{2(1-\cos^22\theta)}{\cos2\theta.\sin2\theta}$$

$$=2\cdot\frac{\dfrac{1}{\cos2\theta}-\dfrac{\cos^22\theta}{\cos2\theta}}{\sin2\theta}=2\cdot\frac{\sec2\theta-\cos2\theta}{\sin2\theta}.$$

(7) $\cot^2\theta - \tan^2\theta = \dfrac{\cos^4\theta - \sin^4\theta}{\cos^2\theta \,.\, \sin^2\theta} = \dfrac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta \,.\, \sin^2\theta}$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta \,.\, \sin^2\theta} = \frac{\cos 2\theta}{\cos^2\theta \,.\, \sin^2\theta} = \frac{4\cos 2\theta}{4\cos^2\theta \,.\, \sin^2\theta} = \frac{4\cos 2\theta}{\sin^2 2\theta}$$

$$= \frac{8\cos 2\theta}{2\sin^2 2\theta} = \frac{8\cos 2\theta}{1 - \cos 4\theta}.$$

(8)

$2\sin A \,.\, \cos 2A = 2\sin A(1 - 2\sin^2 A) = 2\sin A - 4\sin^3 A = \sin 3A - \sin A.$

(9) $\dfrac{\cos nA - \cos(n+2)A}{\sin(n+2)A - \sin nA} = \dfrac{2\sin(n+1)A \,.\, \sin A}{2\cos(n+1)A \,.\, \sin A} = \tan(n+1)A.$

(10)

$$\begin{aligned}\cos 9A + 3\cos 7A + 3\cos 5A + \cos 3A &= \cos 9A + \cos 3A + 3(\cos 7A + \cos 5A)\\ &= 2\cos 6A \,.\, \cos 3A + 6\cos 6A \,.\, \cos A\\ &= 2\cos 6A(\cos 3A + 3\cos A)\\ &= 2\cos 6A \,.\, 4\cos^3 A = 8\cos^3 A \,.\, \cos 6A.\end{aligned}$$

(11) $\dfrac{\operatorname{cosec} 2A - \cot 2A}{\operatorname{cosec} 2A + \cot 2A} = \dfrac{1 - \cos 2A}{1 + \cos 2A} = \dfrac{2\sin^2 A}{2\cos^2 A} = \tan^2 A.$

(12) $\dfrac{1 - \sin A}{1 + \cos A} = \dfrac{\sin^2\frac{A}{2} + \cos^2\frac{A}{2} - 2\sin\frac{A}{2} \cdot \cos\frac{A}{2}}{1 + 2\cos^2\frac{A}{2} - 1} = \dfrac{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)^2}{2\cos^2\frac{A}{2}}$

$$= \frac{1}{2}\left(\frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2}}\right)^2 = \frac{1}{2}\left(1 - \tan\frac{A}{2}\right)^2.$$

(13) $\dfrac{\cos 3A - 2\cos A}{\sin 3A + 2\sin A} \cdot \tan A = \dfrac{4\cos^3 A - 3\cos A - 2\cos A}{3\sin A - 4\sin^3 A + 2\sin A} \cdot \dfrac{\sin A}{\cos A}$

$$= \frac{4\cos^2 A - 3 - 2}{3 - 4\sin^2 A + 2} = \frac{2(2\cos^2 A - 1) - 3}{2(1 - 2\sin^2 A) + 3} = \frac{2\cos 2A - 3}{2\cos 2A + 3}.$$

(14) $$\tan(45^\circ - A) + \tan(45^\circ + A) = \frac{1-\tan A}{1+\tan A} + \frac{1+\tan A}{1-\tan A}$$

$$= \frac{2(1+\tan^2 A)}{1-\tan^2 A} = \frac{2\sec^2 A}{(\cos^2 A - \sin^2 A)\sec^2 A} = \frac{2}{\cos^2 A - \sin^2 A} = 2\sec 2A.$$

(15) $$\cos 2a + \tan\frac{a}{2}\sin 2a = \cos 2a + \frac{\sin\frac{a}{2}}{\cos\frac{a}{2}} \cdot 2\sin a \,.\, \cos a$$

$$= \cos 2a + 4\sin^2\frac{a}{2} \cdot \cos a = 2\cos^2 a - 1 + 4\sin^2\frac{a}{2} \cdot \cos a$$

$$= 2\cos a\left(\cos a + 2\sin^2\frac{a}{2}\right) - 1 = 2\cos a \,.\, 1 - 1 = 2\cos a - 1$$

$$= \cos a + \cos a - 1 = \cos a - 2\sin^2\frac{a}{2} = \cos a - \frac{2 \,.\, \sin^2\frac{a}{2} \cdot \cos\frac{a}{2}}{\cos\frac{a}{2}}$$

$$= \cos a - \tan\frac{a}{2} \cdot \sin a.$$

(16) $$\cot^2 A - \tan^2 A = \frac{\cos^4 A - \sin^4 A}{\cos^2 A \,.\, \sin^2 A} = \frac{(\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)}{\cos^2 A \,.\, \sin^2 A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A \,.\, \sin^2 A} = \frac{4(\cos^2 A - \sin^2 A)}{4\cos^2 A \,.\, \sin^2 A} = \frac{4\cos 2A}{\sin^2 2A} = 4\cot 2A \,.\, \operatorname{cosec} 2A.$$

(17) $$\operatorname{cosec} a \,.\, \cot a - \sec a \,.\, \tan a = \frac{\cos a}{\sin^2 a} - \frac{\sin a}{\cos^2 a} = \frac{\cos^3 a - \sin^3 a}{\sin^2 a \,.\, \cos^2 a}$$

$$= \frac{4(\cos^3 a - \sin^3 a)}{4\sin^2 a \,.\, \cos^2 a} = \frac{4(\cos^3 a - \sin^3 a)}{\sin^2 2a} = 4\operatorname{cosec}^2 2a \,.\, (\cos^3 a - \sin^3 a).$$

(18) $$\cot^2 a - \tan^2 a = \frac{\cos^2 a - \sin^2 a}{\cos^2 a \,.\, \sin^2 a} = \frac{4(\cos^2 a - \sin^2 a)}{4\cos^2 a \,.\, \sin^2 a} = \frac{4\cos 2a}{\sin^2 2a}.$$

(19) $$\operatorname{cosec}^2 b - \sec^2 b = \frac{1}{\sin^2 b} - \frac{1}{\cos^2 b} = \frac{\cos^2 b - \sin^2 b}{\sin^2 b \,.\, \cos^2 b} = \frac{4(\cos^2 b - \sin^2 b)}{4\sin^2 b \,.\, \cos^2 b}$$

$$= \frac{4\cos 2b}{\sin^2 2b} = 4\cos 2b \,.\, \operatorname{cosec}^2 2b.$$

(20) $$\frac{2\,\text{cosec}2A-\sec A}{2\,\text{cosec}2A+\sec A}=\frac{2-\sec A\,.\,\sin 2A}{2+\sec A\,.\,\sin 2A}=\frac{2-2\sin A}{2+2\sin A}=\frac{1-\sin A}{1+\sin A}$$

$$=\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}-2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\left(\frac{\cos\frac{A}{2}-\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}}\right)^2$$

$$=\left(\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}\right)^2=\cot^2\left(45^\circ+\frac{A}{2}\right).$$

(21) $$\sin\left(\frac{5\pi}{2}+\theta\right)-\sin\left(\frac{3\pi}{2}-\theta\right)=2\cos 2\pi\,.\,\sin\left(\frac{\pi}{2}+\theta\right)$$

$$=2\cos 2\pi\,.\,\cos\theta=2\cos 2\pi\,.\,\sin\left(\frac{\pi}{2}-\theta\right)$$

$$=\sin\left(\frac{5\pi}{2}-\theta\right)-\sin\left(\frac{3\pi}{2}+\theta\right).\qquad\text{(Art. 122.)}$$

(22) $$\cot\left(\frac{\pi}{2}+\theta\right)-\tan\left(\frac{\pi}{2}+\theta\right)=\frac{\cos^2\left(\frac{\pi}{2}+\theta\right)-\sin^2\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right).\cos\left(\frac{\pi}{2}+\theta\right)}$$

$$=\frac{\sin^2\theta-\cos^2\theta}{-\cos\theta\,.\,\sin\theta}=\frac{\cos^2\theta-\sin^2\theta}{\sin\theta\,.\,\cos\theta}=\frac{2\,.\,\cos 2\theta}{\sin 2\theta}=2\cot 2\theta.$$

(23) $$\frac{(\text{cosec}\,a+\sec a)^2}{\text{cosec}^2a+\sec^2a}=\frac{\left(\frac{\cos a+\sin a}{\sin a\,.\,\cos a}\right)^2}{\frac{1}{\sin^2a\,.\,\cos^2a}}=(\cos a+\sin a)^2=1+2\sin a\,.\,\cos a$$

$$=1+\sin 2a.$$

(24) $$\frac{\tan\theta}{\tan 2\theta-\tan\theta}=\frac{\tan\theta}{\frac{2\tan\theta}{1-\tan^2\theta}-\tan\theta}=\frac{1}{\frac{2}{1-\tan^2\theta}-1}=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$=\cos^2\theta-\sin^2\theta=\cos 2\theta.$$

(25) $$\frac{\tan 2\theta\,.\,\tan\theta}{\tan 2\theta-\tan\theta}=\frac{\frac{2\tan\theta}{1-\tan^2\theta}\cdot\tan\theta}{\frac{2\tan\theta}{1-\tan^2\theta}-\tan\theta}=\frac{\frac{2\tan\theta}{1-\tan^2\theta}}{\frac{2}{1-\tan^2\theta}-1}=\frac{2\tan\theta}{1+\tan^2\theta}$$

$$=\frac{2\tan\theta}{\sec^2\theta}=2\sin\theta\,.\,\cos\theta=\sin 2\theta.$$

(26) $$\frac{\sin^2\alpha-\sin^2\beta}{\sin\alpha.\cos\alpha-\sin\beta.\cos\beta}=\frac{\sin(\alpha+\beta).\sin(\alpha-\beta)}{\sin\alpha.\cos\alpha-\sin\beta.\cos\beta}$$

(Ex. XXVII. 1.)

$$=\frac{2\sin(\alpha+\beta).\sin(\alpha-\beta)}{2\sin\alpha.\cos\alpha-2\sin\beta.\cos\beta}=\frac{2\sin(\alpha+\beta).\sin(\alpha-\beta)}{\sin2\alpha-\sin2\beta}$$

$$=\frac{2\sin(\alpha+\beta).\sin(\alpha-\beta)}{2\cos(\alpha+\beta).\sin(\alpha-\beta)}=\tan(\alpha+\beta).$$

(27) $$4\sin A.\sin(60^\circ+A).\sin(60^\circ-A)=4\sin A.(\sin^2 60^\circ-\sin^2 A).$$

(Ex. XXVII. 1.)

$$=4\sin A\left(\frac{3}{4}-\sin^2 A\right)=3\sin A-4\sin^3 A=\sin3A.$$

(28) $$\operatorname{cosec}2\theta+\cot4\theta+\operatorname{cosec}4\theta=\frac{1}{\sin2\theta}+\frac{\cos4\theta}{\sin4\theta}+\frac{1}{\sin4\theta}$$

$$=\frac{2\cos2\theta+\cos4\theta+1}{2\sin2\theta.\cos2\theta}=\frac{2\cos2\theta+2\cos^2 2\theta}{2\sin2\theta.\cos2\theta}=\frac{1+\cos2\theta}{\sin2\theta}$$

$$=\frac{2\cos^2\theta}{2\sin\theta.\cos\theta}=\frac{\cos\theta}{\sin\theta}=\cot\theta.$$

2. (1) $$\sin2\theta+\sqrt{3}.\cos2\theta=1,$$

$$\sqrt{3}.\cos2\theta=1-\sin2\theta,$$

$$3.\cos^2 2\theta=1-2\sin2\theta+\sin^2 2\theta,$$

$$3-3\sin^2 2\theta=1-2\sin2\theta+\sin^2 2\theta.$$

Solving this quadratic, we obtain $\sin2\theta=-\frac{1}{2}$, or, 1;

$$\therefore\ 2\theta=-30^\circ, \text{ or, } 90^\circ;$$

$$\therefore\ \theta=-15^\circ, \text{ or, } 45^\circ.$$

(2) $$\sin^2 2\theta-\sin^2\theta=\sin^2\frac{\pi}{4},$$

$$4\sin^2\theta.\cos^2\theta-\sin^2\theta=\frac{1}{2},$$

$$4\sin^2\theta-4\sin^4\theta-\sin^2\theta=\frac{1}{2}.$$

Solving this quadratic, we obtain $\sin^2\theta=\frac{1}{2}$, or, $\frac{1}{4}$;

$$\therefore\ \sin\theta=\frac{1}{\sqrt{2}}, \text{ or, } \frac{1}{2};$$

$$\therefore\ \theta=45^\circ, \text{ or, } 30^\circ.$$

(3) $$\sin 5x \,.\, \cos 3x = \sin 9x \,.\, \cos 7x;$$
$$\therefore \sin 8x + \sin 2x = \sin 16x + \sin 2x;$$
$$\therefore \sin 8x = \sin 16x,$$
$$\sin 8x = 2\sin 8x \,.\, \cos 8x.$$
Hence $\sin 8x = 0$, or, $2\cos 8x = 1$,
$$\sin 8x = 0, \text{ or, } \cos 8x = \frac{1}{2};$$
$$\therefore x = 0^\circ, \text{ or, } 8x = 60^\circ, \text{ and } \therefore x = 7\frac{1}{2}^\circ.$$

(4) $$2\sin^2 3\theta + \sin^2 6\theta = 2,$$
$$\sin^2 6\theta = 2(1 - \sin^2 3\theta),$$
$$4\sin^2 3\theta \,.\, \cos^2 3\theta = 2\cos^2 3\theta,$$
$$2\sin 3\theta \,.\, \cos 3\theta = \sqrt{2}\cos 3\theta.$$
Hence $\cos 3\theta = 0$, or, $\sin 3\theta = \frac{1}{\sqrt{2}}$;
$$\therefore 3\theta = 90^\circ, \text{ or, } 3\theta = 45^\circ;$$
$$\therefore \theta = 30^\circ, \text{ or, } 15^\circ.$$

(5) $$\cos 2A + \sin^2 A = \frac{3}{4}$$
$$1 - 2\sin^2 A + \sin^2 A = \frac{3}{4},$$
$$\sin^2 A = \frac{1}{4}, \text{ and } \therefore \sin A = \pm\frac{1}{2}.$$
Hence $A = 30^\circ$, or, 150°.

(6) $$\cos 3\theta - \cos 5\theta = \sin\theta,$$
$$2\sin 4\theta \,.\, \sin\theta = \sin\theta.$$
Hence $\sin\theta = 0$, or, $\sin 4\theta = \frac{1}{2}$;
$$\therefore \theta = 0^\circ, \text{ or, } 4\theta = 30^\circ;$$
$$\therefore \theta = 0^\circ, \text{ or, } \theta = 7\frac{1}{2}^\circ.$$

(7) $$\sin 5\theta - \cos 3\theta = \sin\theta,$$
$$\sin 5\theta - \sin\theta = \cos 3\theta,$$
$$2\cos 3\theta \,.\, \sin 2\theta = \cos 3\theta.$$
Hence $\cos 3\theta = 0$, or, $\sin 2\theta = \frac{1}{2}$;
$\therefore\ 3\theta = 90^\circ$, or, $2\theta = 30^\circ$,
$\therefore\ \theta = 30^\circ$, or, $\theta = 15^\circ$.

(8) $$\tan 2a = 3\tan a,$$
$$\frac{2\tan a}{1 - \tan^2 a} = 3\tan a.$$
Hence $\tan a = 0$, and $\therefore\ a = 0^\circ$,
or, $2 = 3 - 3\tan^2 a$,
$\tan^2 a = \frac{1}{3}$, or, $\tan a = \frac{1}{\sqrt{3}}$, or, $a = 30^\circ$.

(9) $$\sin 2\theta + \sin\theta = \cos 2\theta + \cos\theta,$$
$$2\sin\frac{3\theta}{2}\cdot\cos\frac{\theta}{2} = 2\cos\frac{3\theta}{2}\cdot\cos\frac{\theta}{2}.$$
$\therefore\ \cos\frac{\theta}{2} = 0$, or, $\frac{\theta}{2} = 90^\circ$, or, $\theta = 180^\circ$;
or, $\sin\frac{3\theta}{2} = \cos\frac{3\theta}{2}$, or, $\tan\frac{3\theta}{2} = 1$, or, $\frac{3\theta}{2} = 45^\circ$, or, $\theta = 30^\circ$.

(10) $$\sin 7a - \sin a = \sin 3a,$$
$$2\cos 4a \,.\, \sin 3a = \sin 3a.$$
$$\text{Hence } \sin 3a = 0, \text{ or, } \left.\begin{array}{l} 3a = 0^\circ, \text{ or, } a = 0^\circ \\ \text{or, } 3a = 180^\circ, \text{ or, } a = 60^\circ \end{array}\right\},$$
or, $2\cos 4a = 1$, or, $4a = 60^\circ$, or, $a = 15^\circ$.

(11) $$\operatorname{cosec}^2\theta - \sec^2\theta = 2\operatorname{cosec}^2\theta \div 3,$$
$$\frac{\operatorname{cosec}^2\theta}{3} = \sec^2\theta, \text{ or, } \cos^2\theta = 3\sin^2\theta;$$
$\therefore\ 4\sin^2\theta = 1$, or, $\sin\theta = \frac{1}{2}$, and $\therefore\ \theta = 30^\circ$.

(12) $$\sin 6\theta = \sin 4\theta - \sin 2\theta,$$

$$\sin 6\theta + \sin 2\theta = \sin 4\theta,$$

$$2\sin 4\theta \,.\, \cos 2\theta = \sin 4\theta.$$

Hence $\sin 4\theta = 0$, or, $4\theta = 0^\circ$, or, $\theta = 0^\circ$,

or, $2\cos 2\theta = 1$, or, $\cos 2\theta = \dfrac{1}{2}$, or, $\theta = 30^\circ$.

EXAMPLES—XXXVI. (p. 106).

1. (1) $$\sin 36^\circ = 2\sin 18^\circ.\cos 18^\circ = 2\cdot\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{(10+2\sqrt{5})}}{4}$$
$$= \frac{2\sqrt{(40-8\sqrt{5})}}{16} = \frac{\sqrt{(10-2\sqrt{5})}}{4}.$$

(2) $$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2.\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \frac{6-2\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4}.$$

(3) $$\sin 54^\circ = \cos 36^\circ = \frac{1+\sqrt{5}}{4}.$$

(4) $$\cos 54^\circ = \sin 36^\circ = \frac{\sqrt{(10-2\sqrt{5})}}{4}.$$

(5) $$\sin 72^\circ = \cos 18^\circ = \sqrt{(1-\sin^2 18^\circ)} = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

(6) $$\tan 72^\circ = \frac{\sin 72^\circ}{\cos 72^\circ} = \frac{\cos 18^\circ}{\sin 18^\circ} = \frac{\sqrt{(10+2\sqrt{5})}}{4} \div \frac{\sqrt{5}-1}{4} = \frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}.$$

(7) $$\sin 90^\circ = \sin(18^\circ + 72^\circ) = \sin 18^\circ.\cos 72^\circ + \cos 18^\circ.\sin 72$$
$$= \sin 18^\circ.\sin 18^\circ + \cos 18^\circ.\cos 18^\circ$$
$$= \left(\frac{\sqrt{5}-1}{4}\right)^2 + \left(\frac{\sqrt{(10+2\sqrt{5})}}{4}\right)^2$$
$$= \frac{6-2\sqrt{5}+10+2\sqrt{5}}{16} = \frac{16}{16} = 1.$$

(8) $$\cos 90^\circ = \cos(18^\circ + 72^\circ) = \cos 18^\circ.\cos 72^\circ - \sin 18^\circ.\sin 72^\circ$$
$$= \cos 18^\circ.\cos 72^\circ - \cos 72^\circ.\cos 18^\circ = 0.$$

2. $\sin(36^\circ+A)+\sin(72^\circ-A)-\sin(36^\circ-A)-\sin(72^\circ+A)$

$$=\{\sin(36^\circ+A)-\sin(36^\circ-A)\}-\{\sin(72^\circ+A)-\sin(72^\circ-A)\}$$

$$=2\cos 36^\circ.\sin A-2\cos 72^\circ.\sin A$$

$$=\sin A\{2\cos 36^\circ-2\cos 72^\circ\}=\sin A\left\{\frac{1+\sqrt{5}}{2}-\frac{\sqrt{5}-1}{2}\right\}=\sin A.$$

Also,

$$\{\sin(54^\circ+A)+\sin(54^\circ-A)\}-\{\sin(18^\circ+A)+\sin(18^\circ-A)\}$$

$$=2\sin 54^\circ.\cos A-2\sin 18^\circ.\cos A$$

$$=\cos A\{2\sin 54^\circ-2\sin 18^\circ\}=\cos A\left\{\frac{1+\sqrt{5}}{2}-\frac{\sqrt{5}-1}{2}\right\}=\cos A.$$

EXAMPLES—XXXVII (p. 110).

(1) At $7\frac{1}{2}^\circ$ the cosine is greater than the sine, and both are positive;

$$\therefore\ \cos\frac{A}{2}+\sin\frac{A}{2}=+\sqrt{1+\sin A},$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=+\sqrt{1-\sin A}.$$

(2) At 150° the cosine (negative) is greater than the sine (positive);

$$\therefore\ \cos\frac{A}{2}+\sin\frac{A}{2}=-\sqrt{1+\sin A},$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=-\sqrt{1-\sin A}.$$

(3) $\cos 189^\circ+\sin 189^\circ=-\sqrt{1+\sin 378^\circ}$,

$\cos 189^\circ-\sin 189^\circ=-\sqrt{1-\sin 378^\circ}$;

$$\therefore\ \cos 189^\circ=-\frac{1}{2}\cdot\left\{\sqrt{1+\frac{\sqrt{5}-1}{4}}+\sqrt{1-\frac{\sqrt{5}-1}{4}}\right\}$$

$$=-\frac{1}{2}\cdot\left\{\frac{\sqrt{3+\sqrt{5}}}{2}+\frac{\sqrt{5-\sqrt{5}}}{2}\right\}$$

$$=-\frac{1}{4}\left\{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}\right\},$$

$$\text{and } \sin 189^\circ = \frac{1}{2}\left\{\sqrt{1-\frac{\sqrt{5}-1}{4}} - \sqrt{1+\frac{\sqrt{5}-1}{4}}\right\}$$
$$= \frac{1}{4}\left\{\sqrt{5-\sqrt{5}} - \sqrt{3+\sqrt{5}}\right\}.$$

(4)
$$2\sin 9^\circ.\,44'.\,30'' = \sqrt{1+\frac{1}{3}} - \sqrt{1-\frac{1}{3}}$$
$$= \sqrt{\frac{4}{3}} - \sqrt{\frac{2}{3}} = \frac{2-\sqrt{2}}{\sqrt{3}};$$
$$\therefore \sin 9^\circ.\,44'.\,30'' = \frac{2-\sqrt{2}}{2\sqrt{3}}.$$

(5)
$$\cos 157^\circ.\,30' = -\sqrt{\frac{1+\cos 315^\circ}{2}} = -\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = -\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$
$$= -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

EXAMPLES—XXXVIII. (p. 111).

(1)
$$\sin A = \frac{3}{5} \text{ and } \sin B = \frac{4}{5},$$
$$\cos A = \frac{4}{5} \text{ and } \cos B = \frac{3}{5};$$
$$\therefore \sin(A+B) = \frac{3}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{4}{5} = \frac{25}{25} = 1;$$
$$\therefore A+B = 90^\circ.$$

(2)
$$\tan A = \frac{1}{7};\ \tan B = \frac{1}{3},$$
$$\tan 2B = \frac{2\tan B}{1-\tan^2 B} = \frac{2}{3} \div \left(1-\frac{1}{9}\right) = \frac{2\times 9}{3\times 8} = \frac{3}{4};$$
$$\therefore \tan(A+2B) = \frac{\tan A + \tan 2B}{1-\tan A\,.\,\tan 2B} = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}} = 1;$$
$$\therefore A+2B = 45^\circ.$$

(3) Let $\sin A = \frac{1}{\sqrt{5}}$ and $\cot B = 3$.

Then $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$;

$$\therefore \tan(A+B) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1;$$

$$\therefore A+B = 45°,$$

that is $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = 45°$.

(4) Let A, B, C, D be the four angles whose tangents are

$$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}.$$

Then $\tan\{(A+B)+(C+D)\}$

$$= \frac{\tan(A+B)+\tan(C+D)}{1-\tan(A+B)\,.\,\tan(C+D)}$$

$$= \left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}} + \frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right) \div \left(1 - \frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}} \cdot \frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right)$$

$$= \left(\frac{4}{7}+\frac{3}{11}\right) \div \left(1-\frac{12}{77}\right) = 1;$$

$$\therefore A+B+C+D = 45°.$$

(5) Let $\cot A = \frac{3}{4}$ and $\cot B = \frac{1}{7}$.

Then $\tan A = \frac{4}{3}$ and $\tan B = 7$;

$$\therefore \tan(A+B) = \frac{\frac{4}{3}+7}{1-\frac{28}{3}} = -1;$$

$$\therefore A+B = 135°, \text{ or, } \cot^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7} = 135°.$$

(6) Let $\tan A=\frac{3}{5}$ and $\tan B=\frac{3}{7}$.

$$\text{Then } \tan(A+B)=\frac{\frac{3}{5}+\frac{3}{7}}{1-\frac{9}{35}}=\frac{18}{13};$$

$$\therefore \cot(A+B)=\frac{13}{18}, \text{ or, } A+B=\cot^{-1}\frac{13}{18}.$$

(7) Let $\tan A=x$ and $\tan B=y$.

$$\text{Then } \tan(A-B)=\frac{x-y}{1+xy};$$

$$\therefore \tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}.$$

(8) Let $\sin A=x$ and $\cos B=x$.

Then $\cos A=\sqrt{1-x^2}$ and $\sin B=\sqrt{1-x^2}$;

$$\therefore \sin(A+B)=x.x+\sqrt{1-x^2}.\sqrt{1-x^2}$$
$$=x^2+1-x^2=1;$$

$$\therefore A+B=90^\circ, \text{ or, } \sin^{-1}x+\cos^{-1}x=90^\circ.$$

(9) Let $\sin A=\frac{4}{5}$, $\sin B=\frac{5}{13}$, $\sin C=\frac{16}{65}$;

$$\therefore \cos A=\frac{3}{5},\ \cos B=\frac{12}{13},\ \cos C=\frac{63}{65}.$$

Then $\sin(A+B+C)=\sin(A+B).\cos C+\cos(A+B).\sin C$

$$=(\sin A.\cos B+\cos A.\sin B)\frac{63}{65}+(\cos A.\cos B-\sin A.\sin B)\frac{16}{65}$$

$$=\left(\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}\right)\cdot\frac{63}{65}+\left(\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}\right)\frac{16}{65}$$

$$=\frac{63}{65}\cdot\frac{63}{65}+\frac{16}{65}\cdot\frac{16}{65}=\frac{4225}{4225}=1.$$

$$\therefore A+B+C=90^\circ, \text{ or, } \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}.$$

(10) Let $\tan A=\frac{1}{5}$, and $\tan B=\frac{1}{239}$.

$$\text{Then } \tan(4A-B)=\frac{\tan 4A-\tan B}{1+\tan 4A\,.\,\tan B}$$

$$=\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=1\,;$$

$$\therefore\ 4A-B=45^\circ,\ \text{or,}\ 4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}.$$

EXAMPLES—XXXIX. (p. 120).

$$\begin{array}{lr@{\qquad}lr@{\qquad}lr}
(1) & \bar{3}\cdot1651553 & (2) & \bar{4}\cdot6843785 & (3) & 2\cdot5324716 \\
 & \bar{4}\cdot7505855 & & \bar{5}\cdot6650657 & & 3\cdot6650657 \\
 & 6\cdot6879746 & & 3\cdot8905196 & & \bar{5}\cdot8905196 \\
 & \bar{2}\cdot6150026 & & \bar{3}\cdot4675284 & & \cdot3156215 \\
\cline{2-2}\cline{4-4}\cline{6-6}
 & \bar{1}\cdot2187180 & & \bar{7}\cdot7074922 & & 2\cdot4036784
\end{array}$$

$$\begin{array}{lr@{\qquad}lr@{\qquad}lr}
(4) & 2\cdot483269 & (5) & \bar{2}\cdot352678 & (6) & \bar{5}\cdot349162 \\
 & \bar{3}\cdot742891 & & \bar{5}\cdot428619 & & \bar{3}\cdot624329 \\
\cline{2-2}\cline{4-4}\cline{6-6}
 & 4\cdot740378 & & 2\cdot924059 & & \bar{3}\cdot724833
\end{array}$$

$$\begin{array}{lr@{\qquad}lr@{\qquad}lr}
(7) & \bar{2}\cdot4596721 & (8) & \bar{7}\cdot429683 & (9) & \bar{9}\cdot2843617 \\
 & 3 & & 6 & & 7 \\
\cline{2-2}\cline{4-4}\cline{6-6}
 & \bar{5}\cdot3790163 & & \overline{40}\cdot578098 & & \overline{62}\cdot9905319
\end{array}$$

$$\begin{array}{lr@{\qquad}lr@{\qquad}lr}
(10) & 3\,\big|\,\underline{\bar{6}\cdot3725409} & (11) & 6\,\big|\,\underline{\overline{14}\cdot432962} & (12) & 9\,\big|\,\underline{\bar{4}\cdot53627188} \\
 & \bar{2}\cdot1241803 & & \bar{3}\cdot738827 & & \bar{1}\cdot61514132
\end{array}$$

EXAMPLES—XL. (p. 123).

1. $\log 128 = \log 2^7 = 7 \log 2 = 2\cdot1072100$

$$\log 125 = \log\frac{1000}{8} = \log 1000 - \log 8 = 3 - \log 2^3$$

$$= 3 - 3\log 2 = 3 - \cdot90309000 = 2\cdot0969100$$

$$\log 2500 = \log\frac{10000}{4} = \log 10000 - \log 4 = 4 - 2\log 2$$

$$= 4 - \cdot6020600 = 3\cdot3979400.$$

2. $\log 50 = \log\frac{100}{2} = \log 100 - \log 2 = 2 - \cdot3010300 = 1\cdot6989700$

$$\log \cdot005 = \log\frac{5}{1000} = \log 10 - \log 2 - 3 = -\log 2 - 2 = \bar{3}\cdot6989700$$

$$\log 196 = \log(49 \times 4) = 2\log 7 + 2\log 2 = 2\cdot2922560.$$

3. $\log 6 = \log 3 + \log 2 = \cdot7781513$

$$\log 27 = 3\log 3 = 1\cdot4313639$$

$$\log 54 = \log(27 \times 2) = 3\log 3 + \log 2 = 1\cdot7323939$$

$$\log 576 = \log(9 \times 64) = 2\log 3 + 6\log 2 = 2\cdot7604226.$$

4. $\log 60 = \log(2 \times 3 \times 10) = \log 2 + \log 3 + \log 10 = 1\cdot7781513$

$$\log \cdot03 = \log\frac{3}{100} = \log 3 - 2 = \cdot4771213 - 2 = \bar{2}\cdot4771213$$

$$\log 1\cdot05 = \log\frac{105}{100} = \log\frac{21}{20} = \log 3 + \log 7 - \log 2 - 1 = \cdot0211893$$

$$\log \cdot0000432 = \log\frac{16 \times 27}{10000000} = 4\log 2 + 3\log 3 - 7 = \bar{5}\cdot6354839$$

5. $\log\cdot00075 = \log 75 - 5 = \log 3 + \log 25 - 5 = \log\left(\frac{18}{2}\right)^{\frac{1}{2}} + \log 25 - 5$

$$= \frac{1}{2}\left\{\log 18 - \log 2\right\} + \log 100 - \log 4 - 5$$

$$= \frac{1}{2}\left\{1\cdot2552725 - \cdot3010300\right\} + 2 - \cdot6020600 - 5$$

$$= \cdot4771213 - \cdot6020600 - 3 = \bar{4}\cdot8750613.$$

$$\begin{aligned}\text{Log } 31\cdot5 &= \log(21\times3\times5)-1=\log 21+\log 3+1-\log 2-1\\ &= \log 21+\frac{1}{2}\Big(\log 18-\log 2\Big)-\log 2\\ &= 1\cdot3222193+\cdot4771212-\cdot3010300=1\cdot4983105.\end{aligned}$$

6. $$\log 2=\log\frac{10}{5}=1-\log 5=\cdot3010300.$$

$$\log\cdot064=\log\frac{2^6}{1000}=6\log 2-3=6-6\log 5-3=\bar{2}\cdot8061800$$

$$\begin{aligned}\log\left\{\frac{2^{60}}{5^{20}}\right\}^{\frac{1}{14}} &= \frac{1}{14}\Big(60\log 2-20\log 5\Big)\\ &= \frac{1}{7}\Big(30-30\log 5-10\log 5\Big)=\frac{1}{7}\Big(30-27\cdot9588000\Big)\\ &= \frac{1}{7}\Big(2\cdot0412000\Big)=\cdot2916000.\end{aligned}$$

7. $$\log 5=\log\frac{10}{2}=1-\cdot3010300=\cdot6989700,$$

$$\log\cdot125=\log\frac{5^3}{1000}=3\log 5-3=2\cdot0969100-3=\bar{1}\cdot0969100$$

$$\begin{aligned}\log\left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}} &= \log 5^{\frac{90}{15}}-\log 2^{\frac{40}{15}}=\log 5^6-\log 2^{\frac{8}{3}}\\ &= 6\log 5-\frac{8}{3}\log 2=6\Big(\log 10-\log 2\Big)-\frac{8}{3}\log 2\\ &= 4\cdot1938200-\cdot8027467=3\cdot3910733.\end{aligned}$$

8. $$\left.\begin{aligned}\cdot01&=\frac{1}{100}=\frac{1}{10^2}=10^{-2}\\ 1&=10^0\\ 100&=10^2\end{aligned}\right\}\therefore \text{ the logarithms are } -2,\ 0,\ 2\ ;$$

$$\left.\begin{aligned}\cdot01&=(\cdot01)^1\\ 1&=(\cdot01)^0\\ 100&=\frac{1}{\cdot01}=(\cdot01)^{-1}\end{aligned}\right\}\therefore \text{ the logarithms are } 1,\ 0,\ -1.$$

9. 1593 is greater than 10^3 and less than 10^4; characteristic 3.
1593 is greater than 12^2 and less than 12^3; characteristic 2.

10. $\frac{4^{3y}}{2^{4y}}=8$; $\frac{2^{6y}}{2^{4y}}=2^3$; $2^{2y}=2^3$; $2y=3$.

Hence $y=\frac{3}{2}$ and $x=\frac{9}{2}$.

11. (*a*) $\log 2=\frac{1}{2}\log 4=\cdot 3010300,$

$$\log 25=\log 100-\log 4=2-\cdot 6020600=1\cdot 3979400$$

$$\log 83\cdot 2=\log(80\times 1\cdot 04)=\frac{3}{2}\log 4+\log 10+\log 1\cdot 04$$

$$=\cdot 9030900+1+\cdot 0170333=1\cdot 9201233$$

$$\log(\cdot 625)^{\frac{1}{100}}=\frac{1}{100}\left\{\log 625-\log 1000\right\}=\frac{1}{100}\left\{2\log 25-3\right\}$$

$$=\frac{1}{100}\left\{2\log 100-2\log 4-3\right\}=\frac{1}{100}\left\{4-1\cdot 2041200-3\right\}$$

$$=-\cdot 0020412=\bar{1}\cdot 9979588.$$

(*b*) $\log(1\cdot 04)^{6000}=6000\log 1\cdot 04=6000\times\cdot 0170333$

$=102\cdot 1998000$; $\therefore$ number of digits is 103.

12. (*a*) $\log 5=\frac{1}{2}\log 25=\cdot 6989700$

$$\log 4=2-\log 25=\cdot 6020600$$

$$\log 51\cdot 5=\log 5+\log 10\cdot 3=\cdot 6989700+1\cdot 0128372=1\cdot 7118072$$

$$\log(\cdot 064)^{\frac{1}{100}}=\frac{1}{100}\left\{\log 64-\log 1000\right\}=\frac{1}{100}\left\{3\log 4-3\right\}$$

$$=\frac{1}{100}\left\{1\cdot 8061800-3\right\}=-\cdot 0119382=\bar{1}\cdot 9880618.$$

(*b*) $\log(1\cdot 03)^{600}=600\log 1\cdot 03=600\times\cdot 0128372$

$=7\cdot 7023200$; $\therefore$ number of digits is 8.

13. $\log 7623 = \log(9 \times 121 \times 7) = 2\log 3 + 2\log 11 + \log 7$

$$= \cdot 9542426 + 2\cdot 0827854 + \cdot 8450980 = 3\cdot 8821260$$

$$\log\frac{77}{300} = \log 7 + \log 11 - \log 3 - \log 100$$

$$= \cdot 8450980 + 1\cdot 0413927 - \cdot 4771213 - 2 = \bar{1}\cdot 4093694$$

$$\log\frac{3}{539} = \log 3 - \log 11 - 2\log 7$$

$$= \cdot 4771213 - 1\cdot 0413927 - 1\cdot 6901960 = \bar{3}\cdot 7455326.$$

14. (1)
$$x\log 4096 = \log 8 - x\log 64$$
$$4x\log 8 = \log 8 - 2x\log 8$$
$$4x = 1 - 2x;\ 6x = 1;\ x = \frac{1}{6}.$$

(2)
$$(2\cdot 5)^x = 6\cdot 25 = (2\cdot 5)^2;\ \therefore\ x = 2.$$

(3)
$$(ab)^x = m;\ x\log(ab) = \log m;$$
$$\therefore\ x = \frac{\log m}{\log a + \log b}.$$

(4)
$$x(m\log a + 2\log b) = \log c;$$
$$\therefore\ x = \frac{\log c}{m\log a + 2\log b}.$$

(5)
$$3x\log a + (4 - x)\log b = (2x - 1)\log c$$
$$x(3\log a - \log b - 2\log c) = -4\log b - \log c;$$
$$\therefore\ x = \frac{4\log b + \log c}{2\log c + \log b - 3\log a}.$$

(6)
$$x(\log a + m\log b) = \log c - 3x\log c$$
$$x(\log a + m\log b + 3\log c) = \log c;$$
$$\therefore\ x = \frac{\log c}{\log a + m\log b + 3\log c}.$$

EXAMPLES—XLI. (p. 127).

(1) log 525030 = 5·7201841
log 525020 = 5·7201758
Difference for 10 = ·0000083
∴ 10 : 5 = ·0000083 : what we must add;
∴ we must add ·0000041;
∴ log 52502·5 = 4·7201799.

(2) log 300430 = 5·4777433
log 300420 = 5·4777288
Difference for 10 = ·0000145
∴ 10 : 5 = ·0000145 : what we must add;
∴ we must add ·0000072;
∴ log 300·425 = 2·4777360.

(3) log 32026000 = 7·5055027
log 32025000 = 7·5054891
Difference for 1000 = ·0000136
∴ 1000 : 613 = ·0000136 : what we must add;
∴ we must add ·0000083;
∴ log 32·025613 = 1·5054974.

(4) log 236610 = 5·3740331
log 236600 = 5·3740147
Difference for 10 = ·0000184
∴ 10 : 1 = ·0000184 : what we must add;
∴ we must add ·0000018;
∴ log 236·601 = 2·3740165.

(5) log 675030 = 5·8293231
log 675020 = 5·8293166
Difference for 10 = ·0000065
∴ 10 : 1 = ·0000065 : what we must add ;
∴ we must add ·0000007 (see end of Art. 162) ;
∴ log 67·5021 = 1·8293173.

(6) log 7333600 = 6·8653172
log 7333500 = 6·8653113
Difference for 100 = ·0000059
∴ 100 : 33 = ·0000059 : what we must add ;
∴ we must add ·0000019 ;
∴ log ·007333533 = $\bar{3}$·8653132.

(7) log 6593200 = 6·8190962
log 6593100 = 6·8190897
Difference for 100 = ·0000065
∴ 100 : 71 = ·0000065 : what we must add ;
∴ we must add ·0000046 ;
∴ log ·000006593171 = $\bar{6}$·8190943.

(8) log 340780 = 5·5324741
log 340770 = 5·5324614
Difference for 10 = ·0000127
∴ 10 : 8 = ·0000127 : what we must add ;
∴ we must add ·0000102 ;
∴ log 3407·78 = 3·5324716.

(9) log 390980 = 5·5921545
log 390970 = 5·5921434
Difference for 10 = ·0000111
∴ 10 : 4 = ·0000111 : what we must add ;
∴ we must add ·0000044 ;
∴ log 390974 = 5·5921478.

(10) $\log 2582000 = 6{\cdot}4119562$

$\log 2581900 = \underline{6{\cdot}4119394}$

Difference for $100 = {\cdot}0000168$

$\therefore 100 : 26 := {\cdot}0000168$: what we must add ;

$\therefore$ we must add ·0000044 ;

$\therefore \log 2{\cdot}581926 = {\cdot}4119438.$

EXAMPLES—XLII. (p. 129).

(1) $\log 12955 = 4{\cdot}1124374$

$\log 12954 = \underline{4{\cdot}1124039}$

Difference for $1 = {\cdot}0000335$

$\therefore {\cdot}0000335 : {\cdot}0000271 = 1$: what has to be added ;

$\therefore$ we must add ·8 ;

$\therefore$ 4·112431 is the logarithm of 12954·8.

(2) $\log 46246 = 4{\cdot}6650742$

$\log 46245 = \underline{4{\cdot}6650648}$

Difference for $1 = {\cdot}0000094$

$\therefore {\cdot}0000094 : {\cdot}0000009 = 1$: what has to be added ;

$\therefore$ we must add ·095 ;

$\therefore$ 3·6650657 is the logarithm of 4624·5095.

(3) $\log 34573 = 4{\cdot}5387371$

$\log 34572 = \underline{4{\cdot}5387245}$

Difference for $1 = {\cdot}0000126$

$\therefore {\cdot}0000126 : 0000114 = 1$: what we must add ;

$\therefore$ we must add ·9047 . . ., or, ·91 ;

$\therefore$ 2·5387359 is the logarithm of 345·7291.

(4) log 39376 = 4·5952316
log 39375 = 4·5952206
Difference for 1 = ·0000110
∴ ·0000110 : ·0000076 = 1 : what we must add;
∴ we must add ·69;
∴ 5·5952282 is the logarithm of 393756·9.

(5) log 37160 = 4·5700757
log 37159 = 4·5700640
Difference for 1 = ·0000117
∴ ·0000117 : ·0000062 = 1 : what we must add;
∴ we must add ·529, or, ·53;
∴ 3·5700702 is the logarithm of 3715·953.

(6) log 96462 = 4·9843563
log 96461 = 4·9843518
Difference for 1 = ·0000045
∴ ·0000045 : ·0000024 = 1 : what we must add;
∴ we must add ·5$\dot{3}$;
∴ $\bar{3}$·9843542 is the logarithm of ·00964615$\dot{3}$.

(7) log 25726 = 4·4103723
log 25725 = 4·4103554
Difference for 1 = ·0000169
∴ ·0000169 : ·0000166 = 1 : what must be added;
∴ we must add ·982;
∴ $\bar{7}$·4103720 is the logarithm of ·00000025725982.

(8) log 60196 = 4·7795604
log 60195 = 4·7795532
Difference for 1 = ·0000072
∴ ·0000072 : ·0000029 = 1 : what must be added;
∴ we must add ·4027, or, ·403;
∴ 2·7795561 is the logarithm of 601·95403.

(9) $\log 10906 = 4{\cdot}0376655$
$\log 10905 = 4{\cdot}0376257$
Difference for $1 = {\cdot}0000398$
$\therefore\ {\cdot}0000398 : {\cdot}0000114 = 1$: what must be added;
$\therefore$ we must add ·286;
$\therefore$ 3·0376371 is the logarithm of 1090·5286.

(10) $\log 26202 = 4{\cdot}4183344$
$\log 26201 = 4{\cdot}4183179$
Difference for $1 = {\cdot}0000165$
$\therefore\ {\cdot}0000165 : {\cdot}0000135 = 1$: what must be added;
$\therefore$ we must add ·818;
$\therefore$ 2·4183314 is the logarithm of 262·01818.

EXAMPLES—XLIII. (p. 132).

(1) $\sin 42^\circ.\ 16' = {\cdot}6725821$
$\sin 42^\circ.\ 15' = {\cdot}6723668$
Difference for $1' = {\cdot}0002153$
$\therefore\ 60'' : 16'' = {\cdot}0002153$: what we must *add;*
$\therefore$ we must add ·0000574;
$\therefore\ \sin 42^\circ.\ 15'.\ 16'' = {\cdot}6724242$.

(2) $\sin 72^\circ.\ 15' = {\cdot}9523958$
$\sin 72^\circ.\ 14' = {\cdot}9523071$
Difference for $1' = {\cdot}0000887$
$\therefore\ 60'' : 6'' = {\cdot}0000887$: what we must *add;*
$\therefore$ we must add ·0000088;
$\therefore\ \sin 72^\circ.\ 14'.\ 6'' = {\cdot}9523159$.

(3)

$$\sin 54^\circ.\,36' = \cdot 8151278$$
$$\sin 54^\circ.\,35' = \underline{\cdot 8149593}$$

Difference for $1' = \cdot 0001685$

$\therefore\ 60'' : 45'' = \cdot 0001685$: what we must *add;*

$\therefore$ we must add $\cdot 0001263$;

$\therefore\ \sin 54^\circ.\,35'.\,45'' = \cdot 8150856.$

(4)

$$\sin 87^\circ.\,27' = \cdot 9990098$$
$$\sin 87^\circ.\,26' = \underline{\cdot 9989968}$$

Difference for $1' = \cdot 0000130$

$\therefore\ 60'' : 15'' = \cdot 0000130$: what we must *add;*

$\therefore$ we must add $\cdot 0000032$;

$\therefore\ \sin 87^\circ.\,26'.\,15'' = \cdot 9990000.$

(5)

$$\sin 43^\circ.\,15' = \cdot 6851830$$
$$\sin 43^\circ.\,14' = \underline{\cdot 6849711}$$

Difference for $1' = \cdot 0002119$

$\therefore\ 60'' : 20'' = \cdot 0002119$: what we must *add;*

$\therefore$ we must add $\cdot 0000706$;

$\therefore\ \sin 43^\circ.\,14'.\,20'' = \cdot 6850417.$

(6)

$$\cos 41^\circ.\,13' = \cdot 7522233$$
$$\cos 41^\circ.\,14' = \underline{\cdot 7520316}$$

Difference for $1' = \cdot 0001917$

$\therefore\ 60'' : 26'' = \cdot 0001917$: what we must *subtract;*

$\therefore$ we must subtract $\cdot 0000830$;

$\therefore\ \cos 41^\circ.\,13'.\,26'' = \cdot 7521403.$

(7)

$$\tan 1^\circ.\,23' = \cdot 0241484$$
$$\tan 1^\circ.\,22' = \underline{\cdot 0238573}$$

Difference for $1' = \cdot 0002911$

$\therefore\ 60'' : 30'' = \cdot 0002911$: what we must *add;*

$\therefore$ we must add $\cdot 0001455$;

$\therefore\ \tan 1^\circ.\,22'.\,30'' = \cdot 0240028.$

(8)
$$\begin{aligned}\cot 35^\circ.\,6' &= 1\cdot4228561\\ \cot 35^\circ.\,7' &= \underline{1\cdot4219766}\\ \text{Difference for } 1' &= \;\;\cdot0008795\end{aligned}$$

$\therefore$ $60'' : 23'' = \cdot0008795$: what we must subtract;

$\therefore$ we must subtract $\cdot0003371$;

$\therefore$ $\cot 35^\circ.\,6'.\,23'' = 1\cdot4225190$.

(9)
$$\begin{aligned}\sin 67^\circ.\,23' &= \cdot9230984\\ \sin 67^\circ.\,22' &= \underline{\cdot9229865}\\ \text{Difference for } 1' &= \cdot0001119\end{aligned}$$

$\therefore$ $60'' : 48''\cdot5 = \cdot0001119$: what we must *add;*

$\therefore$ we must add $\cdot0000904$;

$\therefore$ $\sin 67^\circ.\,22'.\,48''\cdot5 = \cdot9230769$.

(10)
$$\begin{aligned}\cos 34^\circ.\,12' &= \cdot8270806\\ \cos 34^\circ.\,13' &= \underline{\cdot8269170}\\ \text{Difference for } 1' &= \cdot0001636\end{aligned}$$

$\therefore$ $60'' : 19''\cdot6 = \cdot0001636$: what we must *subtract;*

$\therefore$ we must subtract $\cdot0000534$;

$\therefore$ $\cos 34^\circ.\,12'.\,19''\cdot6 = \cdot8270272$.

EXAMPLES—XLIV. (p. 135).

(1)
$$\begin{aligned}\sin 48^\circ.\,47' &= \cdot7522233\\ \sin 48^\circ.\,46' &= \underline{\cdot7520316}\\ \text{Difference for } 1' &= \cdot0001917\end{aligned}$$

$\therefore$ $\cdot0001917 : \cdot0001084 = 60''$: what we must *add* to $48^\circ.\,46'$;

$\therefore$ we must add $34''$;

$\therefore$ the angle is $48^\circ.\,46'.\,34''$.

(2) $\cos 2^\circ.\,33' = \cdot 9990098$

$\cos 2^\circ.\,34' = \cdot 9989968$

Difference for $1' = \cdot 0000130$

$\therefore\ \cdot 0000130 : \cdot 0000098 = 60'' :$ what we must add to $2^\circ.\,33'$;

$\therefore$ we must add $45''$;

$\therefore$ the angle is $2^\circ.\,33'.\,45''$.

(3) $\sin 43^\circ.\,15' = \cdot 6851830$

$\sin 43^\circ.\,14' = \cdot 6849711$

Difference for $1' = \cdot 0002119$

$\therefore\ \cdot 0002119 : \cdot 0000289 = 60'' :$ what we must add to $43^\circ.\,14'$;

$\therefore$ we must add $8''\cdot 18$;

$\therefore$ the angle is $43^\circ.\,14'.\,8''\cdot 18$.

(4) $\cos 32^\circ.\,31' = \cdot 8432351$

$\cos 32^\circ.\,32' = \cdot 8430787$

Difference for $1' = \cdot 0001564$

$\therefore\ \cdot 0001564 : \cdot 0000351 = 60'' :$ what we must add to $32^\circ.\,31'$;

$\therefore$ we must add $13''\cdot 46$, or, approximately, $13''\cdot 5$;

$\therefore$ the angle is $32^\circ.\,31'.\,13''\cdot 5$.

(5) $\sin 24^\circ.\,12' = \cdot 4099230$

$\sin 24^\circ.\,11' = \cdot 4096577$

Difference for $1' = \cdot 0002653$

$\therefore\ \cdot 0002653 : \cdot 0000982 = 60'' :$ what we must add to $24^\circ.\,11'$;

$\therefore$ we must add $22''\cdot 2$;

$\therefore$ the angle is $24^\circ.\,11'.\,22''\cdot 2$.

(6) $\sec 82^\circ.\,23' = 7\cdot 552169$

$\sec 82^\circ.\,22' = 7\cdot 528249$

Difference for $1' = \ \cdot 023920$

$\therefore\ \cdot 023920 : \cdot 005084 = 60'' :$ what we must add to $82^\circ.\,22'$;

$\therefore$ we must add $12''\cdot 8$ nearly;

$\therefore$ the angle is $82^\circ.\,22'.\,12''\cdot 8$.

(7) $\cos 53^\circ.\ 7' = \cdot 6001876$

$\cos 53^\circ.\ 8' = \underline{\cdot 5999549}$

Difference for $1' = \cdot 0002327$

$\therefore \cdot 0002327 : \cdot 0001876 = 60'' :$ what we must add to $53^\circ.\ 7'$;

$\therefore$ we must add $48''\cdot 4$ nearly;

$\therefore$ the angle is $53^\circ.\ 7'.\ 48''\cdot 4$.

(8) $\operatorname{cosec} 25^\circ.\ 3' = 2\cdot 36179$

$\operatorname{cosec} 25^\circ.\ 4' = \underline{2\cdot 36029}$

Difference for $1' = \cdot 00150$

$\therefore \cdot 00150 : \cdot 00068 = 60'' :$ what we must add to $25^\circ.\ 3'$;

$\therefore$ we must add $27''\cdot 2$;

$\therefore$ the angle is $25^\circ.\ 3'.\ 27''\cdot 2$.

(9) $\sin 73^\circ.\ 45' = \cdot 9600499$

$\sin 73^\circ.\ 44' = \underline{\cdot 9599684}$

Difference for $1' = \cdot 0000815$

$\therefore \cdot 0000815 : \cdot 0000316 = 60'' :$ what we must add to $73^\circ.\ 44'$;

$\therefore$ we must add $23''\cdot 2$;

$\therefore$ the angle is $73^\circ.44'.\ 23''\cdot 2$.

(10) $\tan 77^\circ.\ 20' = 4\cdot 44942$

$\tan 77^\circ.\ 19' = \underline{4\cdot 44338}$

Difference for $1' = \cdot 00604$

$\therefore \cdot 00604 : 00106 = 60'' :$ what we must add to $77^\circ.\ 19'$,

$\therefore$ we must add $10''\cdot 5$;

$\therefore$ the angle is $77^\circ.\ 19'.\ 10''\cdot 5$.

EXAMPLES—XLV. (p. 138).

(1) $L \sin 55^\circ.\ 34' = 9\cdot 9163406$

$L \sin 55^\circ.\ 33' = \underline{9\cdot 9162539}$

Difference for $1' = \cdot 0000867$

$\therefore 60'' : 54'' = \cdot 0000867 :$ what we have to add;

$\therefore$ we must add $\cdot 0000780$;

$\therefore L \sin 55^\circ.\ 33'.\ 54'' = 9\cdot 9163319$.

(2) $L\sin 29^\circ.\,26' = 9{\cdot}6914445$

$L\sin 29^\circ.\,25' = \underline{9{\cdot}6912205}$

Difference for $1' = {\cdot}0002240$

$\therefore\ 60'' : 2'' = {\cdot}0002240$: what we have to add;

$\therefore$ we must add ${\cdot}0000075$;

$\therefore\ L\sin 29^\circ.\,25'.\,2'' = 9{\cdot}6912280.$

(3) $L\cos 37^\circ.\,28' = 9{\cdot}8996604$

$L\cos 37^\circ.\,29' = \underline{9{\cdot}8995636}$

Difference for $1' = {\cdot}0000968$

$\therefore\ 60'' : 36'' = {\cdot}0000968$: what we have to subtract;

$\therefore$ we must subtract ${\cdot}0000581$;

$\therefore\ L\cos 37^\circ.\,28'.\,36'' = 9{\cdot}8996023.$

(4) $L\sin 54^\circ.\,14' = 9.9092371$

$L\sin 54^\circ.\,13' = \underline{9{\cdot}9091461}$

Difference for $1' = {\cdot}0000910$

$\therefore\ 60'' : 19'' = {\cdot}0000910$: what we have to add;

$\therefore$ we must add ${\cdot}0000288$;

$\therefore\ L\sin 54^\circ.\,13'.\,19'' = 9{\cdot}9091749.$

(5) $L\tan 27^\circ.\,43' = 9{\cdot}7204759$

$L\tan 27^\circ.\,42' = \underline{9{\cdot}7201690}$

Difference for $1' = {\cdot}0003069$

$\therefore\ 60'' : 34'' = {\cdot}0003069$: what we have to add;

$\therefore$ we must add ${\cdot}0001739$;

$\therefore\ L\tan 27''.\,42'.\,34'' = 9{\cdot}7203429.$

(6) $L\tan 5^\circ.\,14' = 8{\cdot}9618659$

$L\tan 5^\circ.\,13' = \underline{8{\cdot}9604728}$

Difference for $1' = {\cdot}0013931$

$\therefore\ 60'' : 23'' = {\cdot}0013931$: what we have to add;

$\therefore$ we must add ${\cdot}0005340$;

$\therefore\ L\tan 5^\circ.\,13'.\,23'' = 8{\cdot}9610068.$

(7)

$$\begin{aligned}
L\cot 3^\circ.\,37' &= 11{\cdot}1992368\\
L\cot 3^\circ.\,38' &= \underline{11{\cdot}1972347}\\
\text{Difference for } 1' &= {\cdot}0020021
\end{aligned}$$

$\therefore\ 60'' : 50'' = {\cdot}0020021$: what we have to subtract ;

$\therefore$ we must subtract ${\cdot}0016684$;

$\therefore\ L\cot 3^\circ.\,37'.\,50'' = 11{\cdot}1975684.$

(8)

$$\begin{aligned}
L\sin 39^\circ.\,26' &= 9{\cdot}8028968\\
L\sin 39^\circ.\,25' &= \underline{9{\cdot}8027431}\\
\text{Difference for } 1' &= {\cdot}0001537
\end{aligned}$$

$\therefore\ 60'' : 10'' = {\cdot}0001537$: what we have to add ;

$\therefore$ we must add ${\cdot}0000256$;

$\therefore\ L\sin 39^\circ.\,25'.\,10'' = 9{\cdot}8027687.$

(9)

$$\begin{aligned}
L\sin 70^\circ.\,35' &= 9{\cdot}9745697\\
L\sin 70^\circ.\,34' &= \underline{9{\cdot}9745252}\\
\text{Difference for } 1' &= {\cdot}0000445
\end{aligned}$$

$\therefore\ 60'' : 17'' = {\cdot}0000445$: what we must add ;

$\therefore$ we must add ${\cdot}0000126$;

$\therefore\ L\sin 70^\circ.\,34'.\,17'' = 9{\cdot}9745378.$

(10)

$$\begin{aligned}
L\cos 88^\circ.\,54' &= 8{\cdot}2832434\\
L\cos 88^\circ.\,55' &= \underline{8{\cdot}2766136}\\
\text{Difference for } 1' &= {\cdot}0066298
\end{aligned}$$

$\therefore\ 60'' : 16'' = {\cdot}0066298$: what we must subtract ;

$\therefore$ we must subtract ${\cdot}0017679$;

$\therefore\ L\cos 88^\circ.\,54'.\,16'' = 8{\cdot}2814755.$

EXAMPLES—XLVI. (p. 140).

(1)

$$\begin{aligned}
L\sin 14^\circ.\,25' &= 9{\cdot}3961499\\
L\sin 14^\circ.\,24' &= \underline{9{\cdot}3956581}\\
\text{Difference for } 1' &= {\cdot}0004918
\end{aligned}$$

$\therefore\ {\cdot}0004918 : {\cdot}0002868 = 60''$: what we have to add ;

$\therefore$ we must add $35''$ nearly ;

$\therefore$ the angle is $14^\circ.\,24'.\,35''$.

(2) $L\sin 54^\circ.\,14' = 9{\cdot}9092371$
$L\sin 54^\circ.\,13' = \underline{9{\cdot}9091461}$

Difference for $1' = {\cdot}0000910$

$\therefore$ $0000910 : {\cdot}0000299 = 60'' :$ what we have to add;

$\therefore$ we must add $19''$;

$\therefore$ the angle is $54^\circ.\,13'.\,19''$.

(3) $L\sin 71^\circ.\,41' = 9{\cdot}9774191$
$L\sin 71^\circ.\,40' = \underline{9{\cdot}9773772}$

Difference for $1' = {\cdot}0000419$

$\therefore$ ${\cdot}0000419 : {\cdot}0000125 = 60'' :$ what we must add;

$\therefore$ we must add $18''$ nearly;

$\therefore$ the angle is $71^\circ.\,40'.\,18''$.

(4) $L\cos 29^\circ.\,25' = 9{\cdot}9400535$
$L\cos 29^\circ.\,26' = \underline{9{\cdot}9399823}$

Difference for $1' = {\cdot}0000712$

$\therefore$ ${\cdot}0000712 : {\cdot}0000023 = 60'' :$ what we must add;

$\therefore$ we must add $2''$ nearly;

$\therefore$ the angle is $29^\circ.\,25'.\,2''$.

(6) $L\tan 30^\circ.\,51' = 9{\cdot}7761947$
$L\tan 30^\circ.\,50' = \underline{9{\cdot}7759077}$

Difference for $1' = {\cdot}0002870$

$\therefore$ ${\cdot}0002870 : {\cdot}0001320 = 60' :$ what we must add;

$\therefore$ we must add $27''{\cdot}6$ nearly;

$\therefore$ the angle is $30^\circ.\,50'.\,27''{\cdot}6$.

(6) $L\cot 86^\circ.\,32' = 8{\cdot}7823199$
$L\cot 86^\circ.\,33' = \underline{8{\cdot}7802218}$

Difference for $1' = {\cdot}0020981$

$\therefore$ ${\cdot}0020981 : {\cdot}0008556 = 60'' :$ what we must add;

$\therefore$ we must add $24''{\cdot}5$ nearly;

$\therefore$ the angle is $86^\circ.\,32'.\,24''{\cdot}5$.

(7) $L\sin 24^\circ.\,9' = 9{\cdot}6118580$
$L\sin 24^\circ.\,8' = \underline{9{\cdot}6115762}$

Difference for $1' = \ {\cdot}0002818$

$\therefore\ {\cdot}0002818 : {\cdot}0002114 = 60''$: what we must add;

$\therefore$ we must add $45''$;

$\therefore$ the angle is $24^\circ.\,8'.\,45''$.

(8) $L\tan 11^\circ.\,40' = 9{\cdot}3148851$
$L\tan 11^\circ.\,39' = \underline{9{\cdot}3142468}$

Difference for $1' = \ {\cdot}0006383$

$\therefore\ {\cdot}0006383 : {\cdot}0005543 = 60''$: what we must add;

$\therefore$ we must add $52''$;

$\therefore$ the angle is $11^\circ.\,39'.\,52''$.

(9) $L\,\mathrm{cosec}\,46^\circ.\,23' = 10{\cdot}1402787$
$L\,\mathrm{cosec}\,46^\circ.\,24' = \underline{10{\cdot}1401584}$

Difference for $1' = \ {\cdot}0001203$

$\therefore\ {\cdot}0001203 : {\cdot}0000220 = 60''$: what we must add;

$\therefore$ we must add $11''$ nearly;

$\therefore$ the angle is $46^\circ.\,23'.\,11''$.

(10) $L\sec 29^\circ.\,55' = 10{\cdot}0621053$
$L\sec 29^\circ.\,54' = \underline{10{\cdot}0620326}$

Difference for $1' = \ {\cdot}0000727$

$\therefore\ {\cdot}0000727 : {\cdot}0000359 = 60''$: what we must add;

$\therefore$ we must add $29''{\cdot}6$ nearly;

$\therefore$ the angle is $29^\circ.\,54'.\,29''{\cdot}6$.

EXAMPLES—XLVII. (p. 149).

(1) $\sin(A+B) = \sin(180^\circ - C) = \sin C$.

(2) $\cos(A+B) = \cos(180^\circ - C) = -\cos C$.

(3) $\sin\dfrac{A+B}{2} = \sin\left(90^\circ - \dfrac{C}{2}\right) = \cos\dfrac{C}{2}$.

(4) $\cos\frac{A+B}{2}=\cos\left(90^\circ-\frac{C}{2}\right)=\sin\frac{C}{2}.$

(5) $\tan\frac{A+B}{2}=\tan\left(90^\circ-\frac{C}{2}\right)=\cot\frac{C}{2}.$

(6) $\cot\frac{A+B}{2}=\cot\left(90^\circ-\frac{C}{2}\right)=\tan\frac{C}{2}.$

EXAMPLES—XLVIII. (p. 150).

1. (1)
$$\begin{aligned}\sin 2A+\sin 2B+\sin 2C&=2\sin(A+B).\cos(A-B)+\sin 2C\\&=2\sin C.\cos(A-B)+2\sin C.\cos C\\&=2\sin C.\{\cos(A-B)+\cos C\}\\&=2\sin C\{\cos(A-B)-\cos(A+B)\}\\&=2\sin C.(2\sin A.\sin B)\\&=4\sin A.\sin B.\sin C.\end{aligned}$$

(2) $\sin(-A+B+C)+\sin(A-B+C)+\sin(A+B-C)$
$$\begin{aligned}&=2\sin C.\cos(A-B)+\sin(A+B).\cos C-\cos(A+B).\sin C\\&=2\sin C.\cos(A-B)+\sin C.\cos C+\cos C.\sin C\\&=2\sin C.\{\cos(A-B)+\cos C\}\\&=2\sin C.\{\cos(A-B)-\cos(A+B)\}\\&=4\sin A.\sin B.\sin C.\end{aligned}$$

(3)
$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}=\frac{\dfrac{\cos\frac{A}{2}\cdot\sin\frac{C}{2}+\sin\frac{A}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\sin\frac{C}{2}}}{\dfrac{\cos\frac{B}{2}\cdot\sin\frac{C}{2}+\sin\frac{B}{2}\cdot\cos\frac{C}{2}}{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}}=\frac{\sin\left(\frac{A}{2}+\frac{C}{2}\right)\cdot\sin\frac{B}{2}}{\sin\left(\frac{B}{2}+\frac{C}{2}\right)\cdot\sin\frac{A}{2}}$$

$$=\frac{\cos\frac{B}{2}\cdot\sin\frac{B}{2}}{\cos\frac{A}{2}\cdot\sin\frac{A}{2}}=\frac{2\cos\frac{B}{2}\cdot\sin\frac{B}{2}}{2\cos\frac{A}{2}\cdot\sin\frac{A}{2}}=\frac{\sin B}{\sin A}.$$

(4) $\tan(A+B+C)=\tan 180^\circ=0$;

$$\therefore \frac{\tan A+\tan B+\tan C-\tan A\,.\,\tan B\,.\,\tan C}{1-\tan A\,.\,\tan B-\tan B\,.\,\tan C-\tan C\,.\,\tan A}=0;$$

$$\therefore\ \tan A+\tan B+\tan C-\tan A\,.\,\tan B\,.\,\tan C=0;$$

$$\therefore\ \tan A+\tan B+\tan C=\tan A\,.\,\tan B\,.\,\tan C.$$

(5) As in Example (4),

$$\tan A+\tan B+\tan C=\tan A\,.\,\tan B\,.\,\tan C,$$

and dividing both sides by $\tan A\,.\,\tan B\,.\,\tan C$,

$$\cot B\,.\,\cot C+\cot A\,.\,\cot C+\cot A\,.\,\cot B=1.$$

(6) $$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{\cos\frac{A}{2}\cdot\sin\frac{B}{2}+\sin\frac{A}{2}\cdot\cos\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}.$$

$$=\frac{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\cos\frac{C}{2}\left\{\frac{1}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right\}$$

$$=\cos\frac{C}{2}\left\{\frac{\sin\frac{C}{2}+\sin\frac{A}{2}\cdot\sin\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\right\}$$

$$=\cos\frac{C}{2}\left\{\frac{\cos\left(\frac{A}{2}+\frac{B}{2}\right)+\sin\frac{A}{2}\cdot\sin\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\right\}$$

$$=\frac{\cos\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}.$$

(7)

$$\begin{aligned}1+\cos 2A+\cos 2B+\cos 2C&=1+(2\cos^2 A-1)+2\cos(B+C).\cos(B-C)\\&=2\cos^2 A-2\cos A.\cos(B-C)\\&=-2\cos A.\{\cos(B+C)+\cos(B-C)\}\\&=-2\cos A.2\cos B.\cos C=-4\cos A.\cos B.\cos C.\end{aligned}$$

(8)

$$\begin{aligned}\cos A+\cos B+\cos C&=2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}\\&=2\sin\frac{C}{2}\cdot\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1\\&=2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}+1\\&=2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\cdot\sin\frac{B}{2}+1=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}+1.\end{aligned}$$

(9)

$$\begin{aligned}-\sin 2A+\sin 2B+\sin 2C&=2\sin(B+C).\cos(B-C)-2\sin A.\cos A\\&=2\sin A.\{\cos(B-C)-\cos A\}\\&=2\sin A.\{\cos(B-C)+\cos(B+C)\}\\&=4\sin A.\cos B.\cos C.\end{aligned}$$

(10)

$$\begin{aligned}\sin A+\sin B-\sin C&=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cdot\cos\frac{C}{2}\\&=2\cos\frac{C}{2}\cdot\left\{\cos\frac{A-B}{2}-\sin\frac{C}{2}\right\}\\&=2\cos\frac{C}{2}\cdot\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}\\&=2\cos\frac{C}{2}\cdot 2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\\&=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2}\cdot\end{aligned}$$

(11)

$$\begin{aligned}\sin 2A+\sin 2B-\sin 2C&=2\sin(A+B).\cos(A-B)-2\sin C.\cos C\\&=2\sin C.\{\cos(A-B)-\cos C\}\\&=2\sin C.\{\cos(A-B)+\cos(A+B)\}\\&=4\sin C.\cos A.\cos B.\end{aligned}$$

(12) $\cos A+\cos B-\cos C=2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}-\left(1-2\sin^2\frac{C}{2}\right)$

$$=2\sin\frac{C}{2}\cdot\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cdot\cos\frac{A+B}{2}-1$$

$$=2\sin\frac{C}{2}\cdot\left\{\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}-1$$

$$=4\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}-1.$$

(13) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=\frac{1}{2}\left\{\cos A+1+\cos B+1+\cos C+1\right\}$

$$=\frac{1}{2}\cdot\left\{\cos A+\cos B+\cos C+3\right\}$$

$=\frac{1}{2}\cdot\left\{4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}+1+3\right\}$, as in Ex. 8.

$$=2+2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}.$$

(14) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{1}{2}\cdot\left\{1-\cos A+1-\cos B+1-\cos C\right\}$

$=\frac{1}{2}\cdot\left\{3-4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}-1\right\}$, as in Ex. 8.

$$=1-2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}.$$

2.

(1) $\frac{b+c}{a}=\cot A+\operatorname{cosec}A=\frac{\cos A+1}{\sin A}=\frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\cot\frac{A}{2}.$

(2) $2\operatorname{cosec}2A\,.\,\cot B=\frac{2}{\sin 2A}\cdot\frac{\cos B}{\sin B}=\frac{2\cos B}{2\sin A\,.\,\cos A\,.\,\sin B}$

$$=\frac{\cos B}{\cos B\,.\,\sin B\,.\,\sin B}=\frac{1}{\sin^2 B}=\frac{c^2}{b^2}.$$

(3)
$$2\sin^2\frac{B}{2}=1-\cos B=1-\frac{a}{c}=\frac{c-a}{c};$$
$$\therefore \sin\frac{B}{2}=\sqrt{\left(\frac{c-a}{2c}\right)}.$$

(4)
$$2\cos^2\frac{B}{2}=1+\cos B=1+\frac{a}{c}=\frac{a+c}{c};$$
$$\therefore \cos\frac{B}{2}=\sqrt{\left(\frac{a+c}{2c}\right)}.$$

(5)
$$\frac{\cos 2B-\cos 2A}{\sin 2A}=\frac{\cos^2 B-\sin^2 B-\cos^2 A+\sin^2 A}{2\sin A\,.\,\cos A}$$
$$=\frac{\sin^2 A-\sin^2 B-\sin^2 B+\sin^2 A}{2\sin A\,.\,\cos A}=\frac{2\sin^2 A-2\sin^2 B}{2\sin A\,.\,\cos A}$$
$$=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\tan A-\tan B.$$

(6)
$$\tan 2A-\sec 2B=\frac{2\tan A}{1-\tan^2 A}-\frac{1}{\cos^2 B-\sin^2 B}$$
$$=\frac{2ab}{b^2-a^2}-\frac{c^2}{a^2-b^2}=\frac{2ab+c^2}{b^2-a^2}$$
$$=\frac{2ab+a^2+b^2}{b^2-a^2}=\frac{b+a}{b-a}.$$

(7)
$$(\sin A-\sin B)^2+(\cos A+\cos B)^2$$
$$=\sin^2 A-2\sin A\,.\,\sin B+\sin^2 B+\cos^2 A+2\cos A\,.\,\cos B+\cos^2 B$$
$$=2+2(\cos A\,.\,\cos B-\sin A\,.\,\sin B)$$
$$=2+2\cos(A+B)=2-2\cos C=4\sin^2\frac{C}{2}.$$

(8)
$$\sec 2A=\frac{1}{\cos^2 A-\sin^2 A}=\frac{1}{\frac{b^2-a^2}{c^2}}=\frac{c^2}{b^2-a^2}.$$

(9) $a^3.\cos A + b^3.\cos B = a^3.\frac{b}{c} + b^3.\frac{a}{c} = \frac{ab(a^2+b^2)}{c} = \frac{abc^2}{c} = abc.$

(10)

$$\cot(B-A) + \cot 2\left(A + \frac{C}{2}\right) = \frac{\cos B.\cos A + \sin B.\sin A}{\sin B.\cos A - \cos B.\sin A} + \cot(2A + 90^\circ)$$

$$= \frac{\sin A.\sin B + \sin B.\sin A}{\sin B.\sin B - \sin A.\sin A} - \tan 2A$$

$$= \frac{2ab}{b^2-a^2} - \frac{2\tan A}{1-\tan^2 A} = \frac{2ab}{b^2-a^2} - \frac{2ab}{b^2-a^2} = 0.$$

3. (1) $\frac{\sin A - \sin B}{a-b} = \frac{\frac{a\sin C}{c} - \frac{b\sin C}{c}}{a-b} = \frac{(a-b)\sin C}{(a-b)c} = \frac{\sin C}{c}.$

(2) $\frac{\sin(A-B)}{\sin C} = \frac{\sin(A-B).\sin(A+B)}{\sin C.\sin C} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C} = \frac{a^2-b^2}{c^2}.$

(3) $\frac{a.\sin C}{b - a\cos C} = \frac{a\sin C}{a\cos C + c\cos A - a\cos C} = \frac{a.\sin C}{c.\cos A} = \frac{c.\sin A}{c.\cos A} = \tan A.$

(4) $\frac{c}{a}\cdot\operatorname{cosec} B - \cot B = \frac{c}{a.\sin B} - \frac{\cos B}{\sin B} = \frac{c - a.\cos B}{a.\sin B}$

$$= \frac{b\cos A + a\cos B - a\cos B}{a\sin B} = \frac{b\cos A}{b\sin A} = \cot A.$$

(5) $a + b + c = (b\cos C + c\cos B) + (a\cos C + c\cos A) + (a\cos B + b\cos A)$

$$= (a+b)\cos C + (a+c)\cos B + (b+c)\cos A.$$

(6) $\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{2\sin\frac{C}{2}\cdot\cos\frac{C}{2}} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}};$

$$\therefore (a+b).\sin\frac{C}{2} = c.\cos\frac{A-B}{2}.$$

(7) $\frac{a-b}{c}=\frac{\sin A-\sin B}{\sin C}=\frac{2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cdot\cos\frac{C}{2}}=\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$;

$$\therefore (a-b)\cos\frac{C}{2}=c.\sin\frac{A-B}{2}.$$

(8) $\frac{\tan B}{\tan C}=\frac{\sin B.\cos C}{\sin C.\cos B}=\frac{b.\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c.\left(\frac{a^2+c^2-b^2}{2ac}\right)}=\frac{a^2+b^2-c^2}{a^2-b^2+c^2}.$

(9) $c=a\cos B+b\cos A=a\cos B+\frac{a\sin B}{\sin A}\cdot\cos A=a(\cos B+\sin B.\cot A).$

(10) $2(ab.\cos C+ac.\cos B+bc.\cos A)$

$$=(a^2+b^2-c^2)+(a^2+c^2-b^2)+(b^2+c^2-a^2)=a^2+b^2+c^2.$$

(11) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A.\cos B.\cos C$

$$=\frac{1}{2}\left\{1+\cos 2A+1+\cos 2B+1+\cos 2C+4\cos A.\cos B.\cos C\right\}$$

$$=\frac{1}{2}\left\{3+(-1-4\cos A.\cos B.\cos C)+4\cos A.\cos B.\cos C\right\},$$

by Example XLVIII. 1. (7).

$$=\frac{1}{2}\times 2=1.$$

(12)

$$\frac{a-b}{c}\cdot 2\cos^2\frac{C}{2}=\frac{\sin A-\sin B}{\sin C}\cdot 2\cos^2\frac{C}{2}=\frac{2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}{\sin\frac{C}{2}}\cdot\cos\frac{C}{2}$$

$$=2\sin\frac{A-B}{2}\cdot\sin\frac{A+B}{2}=\cos B-\cos A.$$

(13)

$$\frac{a+b}{c}\cdot 2\sin^2\frac{C}{2}=\frac{\sin A+\sin B}{\sin C}\cdot 2\sin^2\frac{C}{2}=\frac{2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{\cos\frac{C}{2}}\cdot\sin\frac{C}{2}$$

$$=2\cos\frac{A-B}{2}\cdot\cos\frac{A+B}{2}=\cos A+\cos B.$$

(14) $a^2 . \sin A + ab . \sin B + ac . \sin C = a^2 \sin A + b . b \sin A + c . c \sin A$

$$= (a^2 + b^2 + c^2) \sin A.$$

(15) By Art. 184, page 149,

$$\cot\frac{A}{2} = \sqrt{\frac{s . (s-a)}{(s-b)(s-c)}} \text{ and } \cot\frac{B}{2} = \sqrt{\frac{s . (s-b)}{(s-a)(s-c)}};$$

$$\therefore \cot\frac{A}{2} : \cot\frac{B}{2} = s - a : s - b$$

$$= b + c - a : a + c - b.$$

(16)

$$\cot\frac{A}{2} \cdot \cot\frac{B}{2} = \sqrt{\frac{s . (s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s . (s-b)}{(s-a)(s-c)}} = \frac{s}{s-c} = \frac{a+b+c}{a+b-c}.$$

(17) $a \sin(B - C) + b \sin(C - A) + c . \sin(A - B)$

$$= a (\sin B . \cos C - \cos B . \sin C) + b (\sin C . \cos A - \cos C . \sin A)$$
$$+ c (\sin A . \cos B - \cos A . \sin B)$$
$$= \cos C(a \sin B - b \sin A) + \cos B(c \sin A - a \sin C)$$
$$+ \cos A(b \sin C - c \sin B)$$
$$= 0 + 0 + 0 = 0.$$

4. If the sides are in arithmetical progression, so also are the sines of the angles ;

$$\therefore \sin A + \sin C = 2 \sin B,$$

$$\text{or } \sin A + \sin(A + B) = 2 \sin B,$$

$$\text{or } 2 \sin\left(A + \frac{B}{2}\right) \cos\frac{B}{2} = 4 \sin\frac{B}{2} \cdot \cos\frac{B}{2};$$

$$\therefore \sin\left(A + \frac{B}{2}\right) = 2 \sin\frac{B}{2}.$$

5. $(b + c) . AD = b . AD + c . AD$

$$= b . b \sin C + c . c \sin B$$
$$= b^2 \sin C + c^2 \sin B.$$

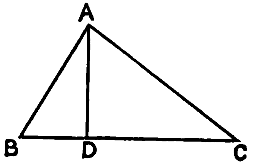


FIG. 20.

6. Let $AB=4, AC=9, BC=12$, and let AD be the line bisecting $\angle BAC$.

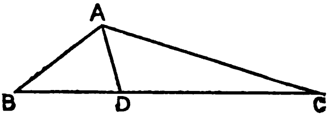


Fig. 21.

Then, by Euclid VI. *B*,

$$BD \,.\, DC + DA^2 = BA \,.\, AC$$

$$AD \cdot \frac{\sin\frac{A}{2}}{\sin B} \times AD \cdot \frac{\sin\frac{A}{2}}{\sin C} + DA^2 = 36$$

$$AD^2\left(\frac{\sin^2\frac{A}{2}}{\sin B \cdot \sin C} + 1\right) = 36$$

$$AD^2\left\{\frac{\frac{(s-b)(s-c)}{bc}}{\frac{4}{a^2bc} \cdot s\,.(s-a).(s-b).(s-c)} + 1\right\} = 36$$

$$AD^2\left\{\frac{a^2}{4.s.(s-a)} + 1\right\} = 36$$

$$AD^2 \times \frac{169}{25} = 36, \text{ or, } AD = \frac{6 \times 5}{13} = 2\tfrac{4}{13}.$$

7. If $\sin A = 2\cos B \,.\, \sin C$

$$\sin(B+C) = 2\cos B \,.\, \sin C$$

$$\sin B \,.\, \cos C + \cos B \,.\, \sin C = 2\cos B \,.\, \sin C$$

$$\sin B \,.\, \cos C - \cos B \sin C = 0$$

$$\sin(B-C) = 0, \text{ and } \therefore B = C.$$

8. If $\cos A\,.\,\cos B\,.\,\sin C = \dfrac{\sin A + \sin B}{\dfrac{\cos A + \cos B}{\cos A\,.\,\cos B}}$

$$\sin C = \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\,.\,\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{2\,.\,\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}};$$

$$\therefore\ 2\sin\frac{C}{2}\cdot\cos\frac{C}{2} = \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}};$$

$$\therefore\ \sin^2\frac{C}{2} = \frac{1}{2},\ \text{or,}\ \sin\frac{C}{2} = \frac{1}{\sqrt{2}};$$

$$\therefore\ \frac{C}{2} = 45^\circ,\ \text{and}\ \therefore\ C = 90^\circ.$$

9.

$$\text{If}\ \sin^2 A = \sin^2 B + \sin^2 C$$

$$\sin^2 A = \frac{b^2}{a^2}\cdot\sin^2 A + \frac{c^2}{a^2}\cdot\sin^2 A;$$

$$\therefore\ a^2 = b^2 + c^2,\ \text{and}\ \therefore\ A = 90^\circ.$$

10.

$$\text{If}\ \frac{\sin A}{\sin C} = \frac{\sin C}{\sin B},\ \text{then}\ \frac{a}{c} = \frac{c}{b},\ \text{or,}\ ab = c^2.$$

$$\text{Then}\ \frac{a^3 + b^3 + c^3}{a + b + c} = ab$$

$$a^3 + b^3 + c^3 = ab(a+b) + abc$$

$$= ab(a+b) + c^3;$$

$$\therefore\ a^3 + b^3 = ab(a+b);$$

$$\therefore\ a^2 - ab + b^2 = ab,\ \text{or,}\ (a-b)^2 = 0,\ \text{or,}\ a = b.$$

Hence a, b, c are all equal.

11.

$$c^2 = a^2 + b^2 - 2ab\,.\,\cos C$$

$$= a^2 + b^2 - 2ab \times \left(-\frac{1}{2}\right)$$

$$= a^2 + b^2 + ab.$$

12. $\frac{\sin A}{\sin B}=\frac{a}{b}$;

$$\therefore \frac{\sin A+\sin B}{\sin A-\sin B}=\frac{a+b}{a-b};$$

$$\therefore \frac{\sin(B+C)+\sin B}{\sin(B+C)-\sin B}=\frac{a+b}{a-b};$$

$$\therefore \frac{\sin\left(B+\frac{C}{2}\right)\cdot\cos\frac{C}{2}}{\cos\left(B+\frac{C}{2}\right)\cdot\sin\frac{C}{2}}=\frac{a+b}{a-b}.$$

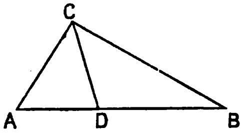


FIG. 22.

Now $\angle ADC=B+\frac{C}{2}$, by EUCLID I. 32;

$$\therefore \tan ADC\,.\cot\frac{C}{2}=\frac{a+b}{a-b};$$

$$\therefore \tan ADC=\frac{a+b}{a-b}\cdot\tan\frac{C}{2}.$$

(13) Draw CE perpendicular to AB.

Then by EUCLID II. XII. and XIII.

$CB^2=CD^2+DB^2+2DB, DE,$

$CA^2=CD^2+DA^2-2AD, DE,$

and $DB=AD$.

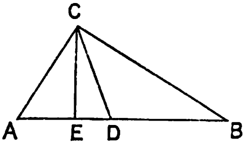


FIG. 23.

$$\therefore\ CB^2+CA^2=2\,CD^2+DB^2+DA^2;$$

$$\therefore\ a^2+b^2=2\,CD^2+\frac{c^2}{4}+\frac{c^2}{4};$$

$$\therefore\ CD^2=\frac{a^2}{2}+\frac{b^2}{2}-\frac{c^2}{4}.$$

EXAMPLES—XLIX. (p. 157).

(1) $a=\sqrt{c^2-b^2}=\sqrt{16}=4$

$$\sin A=\frac{a}{c}=\frac{4}{5}=\cdot 8.$$

Hence, as in Art. 168, we find $A=53^\circ.\ 7'.\ 48''\cdot 4$;

and $\therefore B=36^\circ.\ 52'.\ 11''\cdot 6$.

(2) $$a=\sqrt{c^2-b^2}=\sqrt{64}=8,$$

$$\sin A=\frac{a}{c}=\frac{8}{17}=\cdot4705882.$$

Hence $A=28^\circ.\,4'.\,20''\cdot9$, and $B=61^\circ.\,55'.\,39''\cdot1$.

(3) $$a=\sqrt{c^2-b^2}=\sqrt{400}=20,$$

$$\sin A=\frac{a}{c}=\frac{20}{29}=\cdot6896552.$$

Hence $A=43^\circ.\,36'.\,10''\cdot1$, and $B=46^\circ.\,23'.\,49''\cdot9$.

(4) $$a=\sqrt{c^2-b^2}=\sqrt{576}=24,$$

$$\cos A=\frac{b}{c}=\frac{7}{25}=\cdot28.$$

Hence $A=73^\circ.\,44'.\,23''.\,3$, and $B=16^\circ.\,15'.\,36''\cdot7$.

(5) $$a=\sqrt{c^2-b^2}=\sqrt{3136}=56,$$

$$\cos A=\frac{b}{c}=\frac{33}{65}=\cdot5076923.$$

$\therefore\ A=59^\circ.\,29'.\,23''\cdot2$, and $B=30^\circ.\,30'.\,36''\cdot8$.

(6) $a=c\,.\,\sin A=13\times\cdot9230770=12$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{25}=5$,
$B=22^\circ.\,37'.\,11''\cdot5$.

(7) $a=c\,.\,\sin A=41\times\cdot9756098=40$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{81}=9$,
$B=12^\circ.\,40'.\,49''.\,4$.

(8) $a=c\,.\,\cos B=73\times\cdot6575341=48$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{3025}=55$,
$A=41^\circ.\,6'.\,43''\cdot5$.

(9) $a=c\,.\,\cos B=89\times\cdot4382021=39$ very nearly,
$b=\sqrt{c^2-a^2}=\sqrt{6400}=80$,
$A=25^\circ.\,59'.\,21''\cdot2$.

(10) $b=a\div\tan A=40\div4{\cdot}4444442=9$ very nearly,

$c=\sqrt{a^2+b^2}=\sqrt{1681}=41$,

$B=12^\circ.\ 40'.\ 49''{\cdot}4$.

EXAMPLES—L. (p. 159).

(1) $$b=\sqrt{c^2-a^2}=\sqrt{289\times81}=17\times9=153,$$

$\sin A=\frac{a}{c}$,

$L\sin A=10+2{\cdot}0170333-2{\cdot}2671717=9{\cdot}7498616$;

$\therefore\ A=34^\circ.\ 12'.\ 19''{\cdot}6$, and $B=55^\circ.\ 47'.\ 40''{\cdot}4$.

(2) $$b=\sqrt{c^2-a^2}=\sqrt{729\times121}=27\times11=297,$$

$\sin A=\frac{a}{c}$;

$\therefore\ L\sin A=10+2{\cdot}4828736-2{\cdot}6283889=9{\cdot}8544847$;

$\therefore\ A=45^\circ.\ 40'.\ 2''{\cdot}3$, and $B=44^\circ.\ 19'.\ 57''{\cdot}7$.

(3) $$b=\sqrt{c^2-a^2}=\sqrt{1681\times1}=41,$$

$\sin A=\frac{a}{c}$;

$\therefore\ L\sin A=10+2{\cdot}9242793-2{\cdot}9247960=9{\cdot}9994833$;

$\therefore\ A=87^\circ.\ 12'.\ 20''{\cdot}3$, and $B=2^\circ.\ 47'.\ 39''{\cdot}7$.

(4) $$b=\sqrt{c^2-a^2}=\sqrt{961\times289}=31\times17=527,$$

$\sin A=\frac{a}{c}$;

$\therefore\ L\sin A=10+2{\cdot}5263393-2{\cdot}7958800=9{\cdot}7304593$;

$\therefore\ A=32^\circ.\ 31'.\ 13''{\cdot}5$, and $B=57^\circ.\ 28'.\ 46''{\cdot}5$.

(5) $$b=\sqrt{c^2-a^2}=\sqrt{2209\times9}=47\times3=141,$$

$\sin A=\frac{a}{c}$;

$\therefore\ L\sin A=10+3{\cdot}0413927-3{\cdot}0449315=9{\cdot}9964612$;

$\therefore\ A=82^\circ.\ 41'.\ 44''$, and $B=7^\circ.\ 18'.\ 16''$.

(6)
$$a=\sqrt{c^2-b^2}=\sqrt{968\times 578}\,;$$
$$\therefore \log a=\frac{1}{2}\{\log 968+\log 578\}=2{\cdot}8739016\,;$$
$$\therefore a=748, \text{ and } \cos A=\frac{b}{c}\,;$$
$$\therefore L\cos A=10+2{\cdot}2900346-2{\cdot}8881795=9{\cdot}4018551.$$
$$\therefore A=75^\circ.\ 23'.\ 18''{\cdot}5, \text{ and } B=14^\circ.\ 36'.\ 41''{\cdot}5.$$

(7)
$$a=\sqrt{c^2-b^2}=\sqrt{1058\times 512}\,;$$
$$\therefore \log a=\frac{1}{2}\{\log 1058+9\log 2\}=2{\cdot}8668778\,;$$
$$\therefore a=736, \text{ and } \cos A=\frac{b}{c}\,;$$
$$\therefore L\cos A=10+2{\cdot}4361626-2{\cdot}8948697=9{\cdot}5412929\,;$$
$$\therefore A=69^\circ.\ 38'.\ 56''{\cdot}3, \text{ and } B=20^\circ.\ 21'.\ 3''{\cdot}7.$$

(8)
$$a=\sqrt{c^2-b^2}=\sqrt{1250\times 32}=200,$$
$$\cos A=\frac{b}{c}\,;$$
$$\therefore L\cos A=10+2{\cdot}7846173-2{\cdot}8068580=9{\cdot}9777593\,;$$
$$\therefore A=18^\circ.\ 10'.\ 50'', \text{ and } B=71^\circ.\ 49'.\ 10''.$$

(9)
$$c=\sqrt{a^2+b^2}=\sqrt{76176+243049}=565,$$
$$\tan A=\frac{a}{b}\,;$$
$$\therefore L\tan A=10+2{\cdot}4409091-2{\cdot}6928469=9{\cdot}7480622\,;$$
$$\therefore A=29^\circ.\ 14'.\ 30''{\cdot}3, \text{ and } B=60^\circ.\ 45'.\ 29''{\cdot}7.$$

(10)
$$c=\sqrt{a^2+b^2}=\sqrt{156816+162409}=565,$$
$$\tan A=\frac{a}{b}\,;$$
$$\therefore L\tan A=10+2{\cdot}5976952-2{\cdot}6053050=9{\cdot}9923902\,;$$
$$\therefore A=44^\circ.\ 29'.\ 53'', \text{ and } B=45^\circ.\ 30'.\ 7''.$$

EXAMPLES—LI. (p. 161).

(1) $\dfrac{\text{Height of steeple in feet}}{220} = \tan 46^\circ.\,30'$, and if h be put for height of steeple,

$$\begin{aligned}\log h &= \log 220 + L\tan 46^\circ.\,30' - 10\\ &= 2\cdot 3424227 + \cdot 0227500 = 2\cdot 3651727\,;\\ \therefore\ h &= 231\cdot 835 \text{ feet.}\end{aligned}$$

(2) $\dfrac{BC}{AC} = \tan 25^\circ.\,10'$, and if h be the height of the tower in feet,

$$\frac{h}{200} = \tan 25^\circ.\,10'\,;$$

$$\begin{aligned}\therefore\ \log h &= \log 200 + L.\tan 25^\circ.\,10' - 10\\ &= \log 1000 - \log 5 + 9\cdot 6719628 - 10\\ &= 3 - \cdot 6989700 + 9\cdot 6719628 - 10\\ &= 1\cdot 9729928\,;\\ \therefore\ h &= 93\cdot 97 \text{ feet.}\end{aligned}$$

(3) $BC = 50$ feet ; $\angle BAC = 45^\circ$; $\angle BDC = 30^\circ$.

Then $AC = BC = 50$ feet.

(α) $AD = CD - AC$

$$\begin{aligned}&= BC.\cot 30^\circ - 50\\ &= 50.(\cot 30^\circ - 1) = 50.(\sqrt{3} - 1)\\ &= 50 \times \cdot 7320508 \ldots\\ &= 36\cdot 6025 \ldots \text{ feet.}\end{aligned}$$

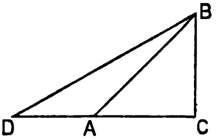


FIG. 24.

(β) $AB = AC.\sec 45^\circ = 50.\sqrt{2} = 50 \times 1\cdot 4142 \ldots = 70\cdot 71 \ldots$ feet.

(γ) $BD = BC.\operatorname{cosec} 30^\circ = 50 \times 2 = 100$ feet.

(4) If h be the measure of the height in feet,

$$\frac{h}{140} = \tan 54^\circ.\,27'\,;$$

$$\therefore\ h = 140 \times 1\cdot 399364 = 195\cdot 910960\,;$$

$\therefore$ height is 196 feet nearly.

(5) Let PC be the hill.

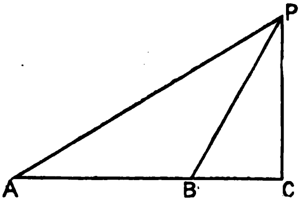


FIG. 25.

Then $\angle PAC = 32°.\ 14'$, and $\angle PBC = 63°.\ 26'$.

Then $PC = BC \tan PBC$,

and $PC = AC . \tan PAC$.

$\therefore BC.\tan PBC = AC . \tan PAC$;

$\therefore BC \times 1\cdot998 = (500 + BC) \times \cdot63$,

whence $BC = 230$ nearly.

Hence $PC = 230 \times 1\cdot998 = 459\cdot54$
$= 460$ yards nearly.

(6) Let θ represent the sun's altitude.

$$\text{Then } \tan\theta = \frac{150}{75} = 2;$$

$$\therefore L\tan\theta = 10 + \log 2 = 10\cdot3010300.$$

$$\text{Hence } \theta = 63°.\ 26'.\ 6''.$$

(7) Let BC be the breadth of the river.

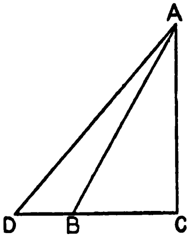


FIG. 26.

Then $AC = BC . \tan 60°$,

and $AC = CD . \tan 50°$.

$\therefore BC . \tan 60° = (40 + BC) \tan 50°$;

$\therefore BC \times \sqrt{3} = (40 + BC) \times 1\cdot19$;

$\therefore BC . (1\cdot73 - 1\cdot19) = 40 \times 1\cdot19$;

$\therefore \cdot54\,BC = 47\cdot6$,

and $\therefore BC = 88$ yards nearly.

(8) Let θ be the angle of inclination.

$$\text{Then } \sin\theta = \frac{60}{109} = \cdot55045.$$

$$\text{Hence } \theta = 33°.\ 23'.\ 55''\cdot7.$$

(9) Let θ be the angle of inclination.

$$\text{Then } \sin\theta = \frac{140}{221} = \cdot6306306;$$

$$\therefore \theta = 39°.\ 5'.\ 47''\cdot9.$$

(10) Let PC be the tower; $\angle PAC = 55°$; $\angle PBC = 48°$.

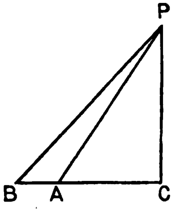


Fig. 27.

$$\text{Then } \frac{PA}{AB} = \frac{\sin 48°}{\sin BPA},$$

$$\text{or } \frac{PA}{30} = \frac{\sin 48°}{\sin 7°};$$

$$\therefore PA = 30 \times \frac{\sin 48°}{\sin 7°},$$

and $AC = PA \cdot \cos PAC = PA \cdot \sin 35°$.

Hence if b be the breadth of the river in feet,

$$b = 30 \times \sin 35° \times \frac{\sin 48°}{\sin 7°}$$

$$\therefore \log b = \log 30 + L\sin 35° + L\sin 48° - L\sin 7° - 10$$

$$= 1{\cdot}47712 + 9{\cdot}75859 + 9{\cdot}87107 - 9{\cdot}08589 - 10$$

$$= 2{\cdot}02089;$$

$$\therefore b = 104{\cdot}93 \text{ feet.}$$

(11) Let AB be the height of the house, BD the length, C the place of observation.

Then ABC and CBD are right angles.

Then $BC = BD \cdot \cot BCD$,

and since $\cos BCD = \frac{1}{\sqrt{5}}$, $\cot BCD = \frac{1}{2}$;

$$\therefore BC = 150 \times \frac{1}{2} = 75 \text{ feet.}$$

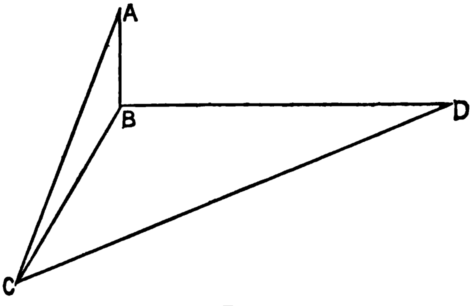


Fig. 28.

Again, $AB=BC\,.\,\tan ACB$,

and since $\sin ACB=\frac{3}{\sqrt{34}}$, $\tan ACB=\frac{3}{5}$;

$$\therefore AB=75\times\frac{3}{5}=45 \text{ feet.}$$

(12) Making the same construction as in Example (11),

$$BC=AB\,.\,\cot ACB=45\times\frac{5}{3}=75 \text{ feet,}$$

and $BD=BC\,.\,\tan BCD=75\times2=150$ feet.

EXAMPLES—LII. (p. 174).

(1) $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{169+1600-1369}{1040}=\frac{5}{13}$;

$$\therefore \sin A=\frac{12}{13}=\cdot9230769.$$

Hence $A=67^\circ.\ 22'.\ 48''\cdot5$.

(2) $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{841+14400-10201}{6960}=\frac{63}{87}$;

$$\therefore \sin A=\frac{60}{87}=\cdot6896552.$$

Hence $A=43^\circ.\ 36'.\ 10''\cdot1$.

(3) $s=\frac{1}{2}(37+13+30)=40$;

$$\therefore \sin\frac{A}{2}=\sqrt{\frac{27\times10}{13\times30}}=\sqrt{\frac{9}{13}};$$

$$\therefore L\sin\frac{A}{2}=10+\frac{1}{2}\left\{\cdot9542425-1\cdot1139434\right\}$$

$$=10-\cdot0798504=9\cdot9201496.$$

Hence $A=112^\circ.\ 37'.\ 11''\cdot5$.

(4) $s=\frac{1}{2}(409+241+600)=625$;

$$\therefore \sin A=\frac{2}{bc}\sqrt{s.(s-a)(s-b)(s-c)}$$
$$=\frac{2}{144600}\sqrt{625\times216\times384\times25}$$
$$=\frac{2\times36000}{144600}=\frac{360}{723};$$
$$\therefore L\sin A=10+2{\cdot}5563025-2{\cdot}8591383=9{\cdot}6971642.$$

Hence $A=29^\circ.\ 51'.\ 46''{\cdot}1$.

2.
$$\frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}}=\frac{c-a}{c+a};$$
$$\therefore \tan\frac{C-A}{2}=\frac{c-a}{c+a}.\cot\frac{B}{2}.$$

Now $c-a=1859$ and $c+a=13419$;

$$\therefore L\tan\frac{C-A}{2}=\log(c-a)-\log(c+a)+L\cot\frac{B}{2};$$
$$\therefore L\tan\frac{C-A}{2}=3{\cdot}26928-4{\cdot}12772+10{\cdot}40312$$
$$=9{\cdot}54468.$$

Hence $\frac{C-A}{2}=19^\circ.\ 18'.\ 50''$.

Also $\frac{C+A}{2}=68^\circ.\ 26'.\ 0''$;

$\therefore C=87^\circ.\ 44'.\ 50''$, and $A=49^\circ.\ 7'.\ 10''$.

3.
$$b=a.\frac{\sin B}{\sin A};$$
$$\therefore \log b=\log a+L\sin B-L\sin A$$
$$=1{\cdot}7403627+9{\cdot}9764927-9{\cdot}8188779$$
$$=1{\cdot}8979775;$$
$$\therefore b=79{\cdot}063.$$

4. $$b=c\cdot\frac{\sin B}{\sin C};$$
$$\therefore \log b=\log c+L\sin B-L\sin C$$
$$=2{\cdot}1613680+9{\cdot}9982047-9{\cdot}8183919$$
$$=2{\cdot}3411808;$$
$$\therefore b=219{\cdot}37.$$

5. $$\sin A=\sin B\cdot\frac{a}{b};$$
$$\therefore L\sin A=L\sin B+\log a-\log b$$
$$=9{\cdot}7175280+2{\cdot}7537623-2{\cdot}5465269$$
$$=9{\cdot}9247634.$$

Hence *one* value of A is 57°. 14′. 21″.

And since a is greater than b, A is greater than B, and we may have the same given parts in a triangle where A is the supplement of 57°. 14′. 21″, or 122°. 45′. 39″.

6. $$\sin B=\frac{b}{c}\cdot\sin C=\frac{16}{8}\cdot\sin 30^\circ=\frac{2}{1}\times\frac{1}{2}=1;$$
$\therefore B=90^\circ$, and the triangle is not ambiguous.

7. In the equilateral triangle $a=b=c$;
$$\therefore \cos A=\frac{a^2+a^2-a^2}{2a^2}=\frac{a^2}{2a^2}=\frac{1}{2}.$$

8. Let $A=60^\circ$, $\frac{b}{c}=\frac{19}{1}$, and $\therefore \frac{b-c}{b+c}=\frac{18}{20}=\frac{9}{10}$.

Now $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}$$
$$=\frac{9}{10}\times\frac{\surd 3}{1}=\frac{3^2\times 3^{\frac{1}{2}}}{10}=\frac{3^{\frac{5}{2}}}{10};$$
$$\therefore L\tan\frac{B-C}{2}=10+\frac{5}{2}\log 3-\log 10$$
$$=10+1{\cdot}1928032-1$$
$$=10{\cdot}1928032;$$
$$\therefore \frac{B-C}{2}=57^\circ.\ 19'.\ 11'',$$
and $$\frac{B+C}{2}=60^\circ.\ 0'.\ 0''.$$
$$\therefore B=117^\circ.\ 19'.\ 11'', \text{ and } C=2^\circ.\ 40'.\ 49''.$$

9. Let a, b, c denote the sides in order of the given values.

$$\text{Then } \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{6+(1+\sqrt{3})^2-4}{2(1+\sqrt{3}).\sqrt{6}} = \frac{6+2\sqrt{3}}{2\sqrt{6}+6\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\therefore\ A = 45°.$$

$$\text{Again, } \sin B = \frac{b}{a}\cdot\sin A = \frac{\sqrt{6}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore\ B = 60°;$$

$$\text{and } \therefore\ C = 180° - (60°+45°) = 180° - 105° = 75°.$$

10. Construct a diagram, as in Art. 213, fig. 2, but with A and B interchanged, because B is here to be the *smaller* angle.

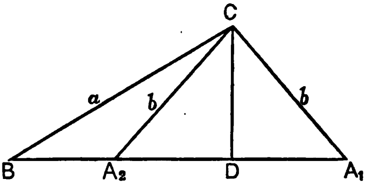


FIG. 29.

Let $c_1 = A_2B$, and $c_2 = A_1B$.

$$\text{Then } c_1 = BD - A_2D = a\cos B - b\,.\cos CA_2D,$$

$$\text{and } c_2 = BD + A_1D = a\cos B + b\,.\cos CA_2D;$$

$$\begin{aligned}\therefore\ c_1\,.\,c_2 &= a^2.\cos^2 B - b^2.\cos^2 CA_2D\\ &= a^2.\cos^2 B - b^2.\cos^2 A\\ &= a^2.(1-\sin^2 B) - b^2.(1-\sin^2 A)\\ &= a^2 - b^2;\end{aligned}$$

$$\therefore\ c_1\,.\,c_2 + b^2 = a^2.$$

11. Let $A = 64°.12'$, and $\dfrac{b}{c} = \dfrac{9}{7}$.

$$\text{Then } \frac{b-c}{b+c} = \frac{9-7}{9+7} = \frac{2}{16} = \frac{1}{8}.$$

$$\begin{aligned}\text{And } \tan\frac{B-C}{2} &= \frac{b-c}{b+c}\cdot\cot\frac{A}{2}\\ &= \frac{1}{8}\cdot\cot 32°.6'.\end{aligned}$$

$$\begin{aligned}\therefore\ L\tan\frac{B-C}{2}&=\log 1-\log 8+L\cot 32^\circ.\,6'\\&=0-3\log 2+L\tan 57^\circ.\,54'\\&=-{\cdot}90309+10{\cdot}2025255\\&=9{\cdot}2994355.\end{aligned}$$

$$\text{Hence }\frac{B-C}{2}=11^\circ.\,16'.\,10'',$$

$$\text{and }\frac{B+C}{2}=57^\circ.\,54'.\,0'';$$

$$\therefore\ B=69^\circ.\,10'.\,10'',\text{ and }C=46^\circ.\,37'.\,50''.$$

12.

$$s=\frac{15}{2},\ s-a=\frac{7}{2},\ s-b=\frac{5}{2},\ s-c=\frac{3}{2}\cdot$$

$$\therefore\ \cos\frac{B}{2}=\sqrt{\frac{15\times 5}{2.2.4.6}}=\sqrt{\frac{25}{2^5}};$$

$$\begin{aligned}\therefore\ L\cos\frac{B}{2}&=10+\frac{1}{2}\left\{2\log 5-5\log 2\right\}\\&=10+\frac{1}{2}\left\{1{\cdot}3979400-1{\cdot}5051495\right\}\\&=9{\cdot}9463953.\end{aligned}$$

$$\text{Hence }\frac{B}{2}=27^\circ.\,53'.\,8'',\text{ and }B=55^\circ.\,46'.\,16''.$$

13.

$$\begin{aligned}\tan\frac{A-B}{2}&=\frac{a-b}{a+b}\cdot\cot\frac{C}{2}\\&=\frac{70-35}{70+35}\cdot\cot\frac{C}{2}\\&=\frac{1}{3}\cot.\,18^\circ.\,26'.\,6'';\end{aligned}$$

$$\begin{aligned}\therefore\ L\tan\frac{A-B}{2}&=\log 1-\log 3+L\cot 18^\circ.\,26'.\,6''\\&=0-{\cdot}4771213+10{\cdot}4771213\\&=10;\end{aligned}$$

$$\therefore\ \frac{A-B}{2}=45^\circ,$$

$$\text{and }\frac{A+B}{2}=71^\circ.\,33'.\,54'';$$

$$\therefore\ A=116^\circ.\,33'.\,54'',\text{ and }B=26^\circ.\,33'.\,54''.$$

EXAMPLES—LIII. (p. 176).

(1) $$c=\sqrt{a^2+b^2}=\sqrt{16+9}=5,$$

$$\sin A=\frac{4}{5}=\cdot 8.$$

By the tables $\sin 53^\circ.\ 7'=\cdot 7998593,$
$\sin 53^\circ.\ 8'=\cdot 8000338.$

Hence $A=53^\circ.\ 7'.\ 48''\cdot 4$, and $B=36^\circ.\ 52'.\ 11''\cdot 6$.

(2) $$a=\sqrt{c^2-b^2}=48,$$

$$\sin B=\frac{55}{73}=\cdot 7535068.$$

By the tables $\sin 48^\circ.\ 53'=\cdot 7533721,$
$\sin 48^\circ.\ 54'=\cdot 7535634.$

Hence $B=48^\circ.\ 53'.\ 16''\cdot 5$, and $A=41^\circ.\ 6'.\ 43''\cdot 5$.

(3) $$c=\sqrt{a^2+b^2}=353,$$

$$\sin A=\frac{272}{353}=\cdot 7705382.$$

By the tables $\sin 50^\circ.\ 24'=\cdot 7705132$
$\sin 50^\circ.\ 25'=\cdot 7706986.$

Hence $A=50^\circ.\ 24'.\ 8''\cdot 1$, and $B=39^\circ.\ 35'.\ 51''\cdot 9$.

(4) $$a=\sqrt{c^2-b^2}=40,$$

$$\sin A=\frac{40}{401}=\cdot 0997506.$$

By the tables $\sin 5^\circ.\ 43'=\cdot 0996092,$
$\sin 5^\circ.\ 44'=\cdot 0998986.$

Hence $A=5^\circ.\ 43'.\ 29''\cdot 3$, and $B=84^\circ.\ 16'.\ 30''\cdot 7$.

(5) $$B=79^\circ.\ 7'.\ 9''\cdot 6.$$

By the tables $\sin 10^\circ.\ 52'=\cdot 1885241,$
$\sin 10^\circ.\ 53'=\cdot 1888098.$

Hence $\sin 10^\circ.\ 52'.\ 50''\cdot 4=\cdot 1887639$;
$\therefore\ a=c\,.\,\sin A=445\times\cdot 1887639=84,$
and $b=\sqrt{c^2-a^2}=437.$

(6) $B=43°.\ 0'.\ 10''\cdot 3.$

By the tables $\sin 46°.\ 59'=\cdot 7311553,$
$\sin 47°.\ 0'=\cdot 7313537.$
Hence $\sin 46°.\ 59'.\ 49''\cdot 7=\cdot 7313196$;
$\therefore\ a=c\,.\,\sin A=629\times\cdot 7313196=460,$
and $b=\sqrt{c^2-a^2}=429.$

(7) $A=38°.\ 34'.\ 48''\cdot 3.$

By the tables $\sin 51°.\ 25'=\cdot 7817019,$
$\sin 51°.\ 26'=\cdot 7818833.$
Hence $\sin 51°.\ 25'.\ 11''\cdot 7=\cdot 7817372$;
$\therefore\ b=c\,.\,\sin B=449\times\cdot 7817372=351,$
and $a=\sqrt{c^2-b^2}=280.$

(8) $A=31°.\ 2'.\ 53''\cdot 6.$

By the tables $\sin 58°.\ 57'=\cdot 8567175,$
$\sin 58°.\ 58'=\cdot 8568675.$
Hence $\sin 58°.\ 57'.\ 6''\cdot 4=\cdot 8567335$;
$\therefore\ b=c\,.\,\sin B=349\times\cdot 8567335=299,$
and $a=\sqrt{c^2-b^2}=180.$

(9) $B=23°.\ 57'.\ 8''.$

By the tables $\tan 23°.\ 57'=\cdot 4441834,$
$\tan 23°.\ 58'=\cdot 4445318.$
Hence $\tan 23°.\ 57'.8''=\cdot 4442365$;
$\therefore\ b=a\,.\,\tan B=520\times\cdot 4442365=231,$
and $c=\sqrt{a^2+b^2}=569.$

(10) $B=3°.\ 41'.\ 43''.$

By the tables $\tan 86°.\ 18'=15\cdot 463814,$
$\tan 86°.\ 19'=15\cdot 533981.$
Hence $\tan 86°.\ 18'.\ 17''=15\cdot 483694$;
$\therefore\ a=b\,.\,\tan A=31\times 15\cdot 483694=480,$
and $c=\sqrt{a^2+b^2}=481.$

EXAMPLES—LIV. (p. 177).

(1) $s=245,\ s-a=48,\ s-b=192,\ s-c=5.$

$$\text{Then } \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s\,.\,(s-a)}};$$

$$\therefore L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 192+\log 5-\log 245-\log 48\right\}$$

$$=10+\frac{1}{2}\left\{2\cdot2833012+\cdot6989700-2\cdot3891661-1\cdot6812412\right\}$$

$$=10+\frac{1}{2}\left\{2\cdot9822712-4\cdot0704073\right\}$$

$$=9\cdot4559320.$$

Hence $\frac{A}{2}=15^\circ.\ 56'.\ 43''\cdot4$, and $\therefore A=31^\circ.\ 53'.\ 26''\cdot8$.

By a similar method we may find $B=8^\circ.\ 10'.\ 16''\cdot4$,

and $\therefore C=139^\circ.\ 56'.\ 16''\cdot8$.

(2.) $s=605,\ s-a=96,\ s-b=384,\ s-c=125.$

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 384+\log 125-\log 605-\log 96\right\}$$

$$=10+\frac{1}{2}\left\{2\cdot5843312+2\cdot0969100-2\cdot7817554-1\cdot9822712\right\}$$

$$=10+\frac{1}{2}\left\{4\cdot6812412-4\cdot7640266\right\}$$

$$=9\cdot9586703.$$

Hence $\frac{A}{2}=42^\circ.\ 16'.\ 25''\cdot25$, and $\therefore A=84^\circ.\ 32'.\ 50''\cdot5$.

By a similar method we find $B=25^\circ.\ 36'.\ 30''\cdot7$,

and $\therefore C=69^\circ.\ 50'\ 38''\cdot8$.

(3) $s=680,\ s-a=147,\ s-b=363,\ s-c=170.$

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 363+\log 170-\log 680-\log 147\right\}$$

$$=10+\frac{1}{2}\left\{2{\cdot}5599066+2{\cdot}2304489-2{\cdot}8325089-2{\cdot}1673173\right\}$$

$$=10+\frac{1}{2}\left\{4{\cdot}7903555-4{\cdot}9998262\right\}$$

$$=9{\cdot}8952647.$$

Hence $\frac{A}{2}=38^\circ.\,9'.\,26''$, and $\therefore\ A=76^\circ.\,18'.\,52.''$

By a similar method we find $B=35^\circ.\,18'.\,0''{\cdot}9$,

and $\therefore\ C=68^\circ.\,23'.\,7''{\cdot}1$.

(4) $s=808,\ s-a=243,\ s-b=363,\ s-c=202.$

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 363+\log 202-\log 808-\log 243\right\}$$

$$=10+\frac{1}{2}\left\{2{\cdot}5599066+2{\cdot}3053514-2{\cdot}9074114-2{\cdot}3856063\right\}$$

$$=10+\frac{1}{2}\left\{4{\cdot}8652580-5{\cdot}2930177\right\}$$

$$=9{\cdot}7861202.$$

Hence $\frac{A}{2}=31^\circ.\,25'.\,46''{\cdot}45$, and $\therefore\ A=62^\circ.\,51'.\,32''{\cdot}9$.

By a similar method we find $B=44^\circ.\,29'.\,53''$,

and $\therefore\ C=72^\circ.\,38'.\,34''{\cdot}1$.

(5) $s=416,\ s-a=7,\ s-b=175,\ s-c=234.$

$$L\tan\frac{A}{2}=10+\frac{1}{2}\left\{\log 175+\log 234-\log 416-\log 7\right\}$$

$$=10+\frac{1}{2}\left\{2{\cdot}2430380+2{\cdot}3692159-2{\cdot}6190933-{\cdot}8450980\right\}$$

$$=10+\frac{1}{2}\left\{4{\cdot}6122539-3{\cdot}4641913\right\}$$

$$=10{\cdot}5740313.$$

Hence $\frac{A}{2}=75^\circ.\,4'.\,7''$, and $\therefore\ A=150^\circ.\,8'.\,14''$.

By a similar method we can find $B=17^\circ.\,3'.\,41''{\cdot}5$,

and $\therefore\ C=12^\circ.\,48'.\,4''{\cdot}5$.

(6)
$$B=180^\circ-(A+C)=11^\circ.\ 25'.\ 16''\cdot 3,$$
$$a=b\,.\,\frac{\sin A}{\sin B}=\frac{29\times\cdot 6896550}{\cdot 1980199}=101,$$
$$c=b\,.\,\frac{\sin C}{\sin B}=\frac{29\times\cdot 8193229}{\cdot 1980199}=120.$$

(7)
$$B=180^\circ-(A+C)=39^\circ.\ 18'.\ 27''\cdot 5,$$
$$a=b\cdot\frac{\sin A}{\sin B}=\frac{149\times\cdot 9395972}{\cdot 6338400}=221,$$
$$c=b\cdot\frac{\sin C}{\sin B}=\frac{149\times\cdot 9438490}{\cdot 6338400}=222.$$

(8)
$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$
$$\tan\frac{A-B}{2}=\frac{72}{130}\cdot\cot 16^\circ.\ 5'.\ 26''\cdot 9,$$
$$L\tan\frac{A-B}{2}=\log 72-\log 130+L\cot 16^\circ.\ 5'.\ 26''\cdot 9$$
$$=1\cdot 8573325-2\cdot 1139434+10\cdot 5399616$$
$$=10\cdot 2833507.$$
$$\text{Hence }\frac{A-B}{2}=62^\circ.\ 29'.\ 16''\cdot 8,$$
$$\text{and }\frac{A+B}{2}=73^\circ.\ 54'.\ 33''\cdot 1\ ;$$
$$\therefore\ A=136^\circ.\ 23'.\ 49''\cdot 9,\text{ and }B=11^\circ.\ 25'.\ 16''\cdot 3.$$
$$\text{Also, }c=\frac{a\,.\,\sin C}{\sin A}=\frac{101\times\cdot 5326047}{\cdot 6896550}=78.$$

(9)
$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$
$$\tan\frac{A-B}{2}=\frac{360}{442}\cot 48^\circ.\ 28'.\ 40''\cdot 05,$$
$$L\tan\frac{A-B}{2}=\log 360-\log 442+L\cot 48^\circ.\ 28'.\ 40''\cdot 05$$
$$=2\cdot 5563025-2\cdot 6454223+9\cdot 9471473$$
$$=9\cdot 8580275.$$

$$\text{Hence } \frac{A-B}{2}=35^\circ.\ 47'.\ 50''\cdot 65,$$

$$\text{and } \frac{A+B}{2}=41^\circ.\ 31'.\ 19''\cdot 95\ ;$$

$$\therefore\ A=77^\circ.\ 19'.\ 10''\cdot 6, \text{ and } B=5^\circ.\ 43'.\ 29''\cdot 2.$$

$$\text{Also, } c=\frac{a\,.\,\sin C}{\sin A}=\frac{401\times\cdot 9926403}{\cdot 9756097}=408.$$

(10)

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{72}{370}\cot 15^\circ.\ 20'.\ 17''\cdot 5,$$

$$L\tan\frac{A-B}{2}=\log 72-\log 370+L\cot 15^\circ.\ 20'.\ 17''\cdot 5$$

$$=1\cdot 8573325-2\cdot 5682017+10\cdot 5617669$$
$$=9\cdot 8508977.$$

$$\text{Hence } \frac{A-B}{2}=35^\circ.\ 21'.\ 15'',$$

$$\text{and } \frac{A+B}{2}=74^\circ.\ 39'.\ 42''\cdot 5\ ;$$

$$\therefore\ A=110^\circ.\ 0'.\ 57''\cdot 5, \text{ and } B=39^\circ.\ 18'.\ 27''\cdot 5.$$

$$\text{Also, } c=\frac{a\,.\,\sin C}{\sin A}=\frac{221\times\cdot 5101885}{\cdot 9395972}=120.$$

(11)

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{48}{170}\cdot\cot 33^\circ.\ 29'.\ 42''\cdot 7,$$

$$L\tan\frac{A-B}{2}=\log 48-\log 170+L\cot 33^\circ.\ 29'.\ 42''\cdot 7$$

$$=1\cdot 6812412-2\cdot 2304489+10\cdot 1792962$$
$$=9\cdot 6300885.$$

$$\text{Hence } \frac{A-B}{2}=23^\circ.\ 6'.\ 57''\cdot 3,$$

$$\text{and } \frac{A+B}{2}=56^\circ.\ 29'.\ 42''\cdot 7\ ;$$

$$\therefore\ A=79^\circ.\ 36'.\ 40'', \text{ and } B=33^\circ.\ 23'.\ 54''\cdot 6.$$

$$\text{Also, } c=\frac{a\,.\,\sin C}{\sin A}=\frac{109\times\cdot 9204413}{\cdot 9836064}=102.$$

(12) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{362}{528}\cdot\cot 43^\circ.\,57'.\,30'',$$

$$L\tan\frac{A-B}{2}=\log 362-\log 528+L\cot 43^\circ.\,57'.\,30''$$

$$=2\cdot5587086-2\cdot7226339+10\cdot0157949$$

$$=9\cdot8518696.$$

$$\text{Hence }\frac{A-B}{2}=35^\circ.\,24'.\,46'',$$

$$\text{and }\frac{A+B}{2}=46^\circ.\,2'.\,30'';$$

$$\therefore A=81^\circ.\,27'.\,16'',\text{ and }B=10^\circ.\,37'.\,44''.$$

$$\text{Also, }c=\frac{b\,.\,\sin C}{\sin B}=\frac{83\times\cdot999390}{\cdot1844460}=450.$$

(13) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cdot\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{120}{338}\cdot\cot 65^\circ.\,42'.\,22'',$$

$$L\tan\frac{A-B}{2}=\log 120-\log 338+L\cot 65^\circ.\,42'.\,22''$$

$$=2\cdot0791812-2\cdot5289167+9\cdot6545508$$

$$=9\cdot2048153.$$

$$\text{Hence }\frac{A-B}{2}=9^\circ.\,6'.\,16''\cdot6,$$

$$\text{and }\frac{A+B}{2}=24^\circ.\,17'.\,38'';$$

$$\therefore A=33^\circ.\,23'.\,54''\cdot6,\text{ and }B=15^\circ.\,11'.\,21''\cdot4.$$

$$\text{Also, }c=\frac{b\,.\,\sin C}{\sin B}=\frac{109\times\cdot7499700}{\cdot2620086}=312.$$

(14) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{72}{410}\cdot\cot 52^\circ.\ 1'.\ 55''{\cdot}5,$$

$$L\tan\frac{A-B}{2}=\log 72-\log 410+L\cot 52^\circ.\ 1'.\ 55''{\cdot}5$$

$$=1{\cdot}8573325-2{\cdot}6127839+9{\cdot}8923085$$

$$=9{\cdot}1368571.$$

$$\therefore\ \frac{A-B}{2}=7^\circ.\ 48'.\ 12'',$$

$$\text{and } \frac{A+B}{2}=37^\circ.\ 58'.\ 4''{\cdot}5\ ;$$

$$\therefore\ A=45^\circ.\ 46'.\ 16''{\cdot}5, \text{ and } B=30^\circ.\ 9'.\ 52''{\cdot}5.$$

$$\text{Also, } c=\frac{b\sin C}{\sin B}=\frac{169\times{\cdot}9900242}{{\cdot}5024855}=332{\cdot}97.$$

(15) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\tan\frac{A-B}{2}=\frac{72}{410}\cot 7^\circ.\ 41'.\ 18''{\cdot}5,$$

$$L\tan\frac{A-B}{2}=\log 72-\log 410+L\cot 7^\circ.\ 41'.\ 18''{\cdot}5$$

$$=1{\cdot}8573325-2{\cdot}6127839+10{\cdot}8696637$$

$$=10{\cdot}1142123.$$

$$\text{Hence } \frac{A-B}{2}=52^\circ.\ 26'.\ 54''{\cdot}1,$$

$$\text{and } \frac{A+B}{2}=82^\circ.\ 18'.\ 41''{\cdot}5\ ;$$

$$\therefore\ A=134^\circ.\ 45'.\ 36''{\cdot}6, \text{ and } B=29^\circ.\ 51'.\ 46''{\cdot}4.$$

$$\text{Also, } c=\frac{b\,.\,\sin C}{\sin B}=\frac{169\times{\cdot}2651681}{{\cdot}4982927}=90.$$

(16) $$\sin B=\frac{b\,.\,\sin A}{a}=\frac{37\times\sin 18^\circ.\,55'.\,28''{\cdot}7}{13};$$

$$\therefore\ L\sin B=\log 37+L\sin 18^\circ.\,55'.\,28''{\cdot}7-\log 13$$
$$=1{\cdot}5682017+9{\cdot}5109783-1{\cdot}1139434$$
$$=9{\cdot}9652366;$$

$\therefore\ B=67^\circ.\,22'.\,48''{\cdot}1$, or its supplement $112^\circ.\,37'.\,11''{\cdot}9$.

(17) $$\sin B=\frac{b\,.\,\sin A}{a}=\frac{565\times\sin 44^\circ.\,29'.\,53''}{445};$$

$$\therefore\ L\sin B=\log 565+L\sin 44^\circ.\,29'.\,53''-\log 445$$
$$=2{\cdot}7520484+9{\cdot}8456468-2{\cdot}6483600$$
$$=9{\cdot}9493352;$$

$\therefore\ B=62^\circ.\,51'.\,32''{\cdot}9$, or its supplement $117^\circ.\,8'.\,27''{\cdot}1$.

(18) $$\sin B=\frac{b\,.\,\sin A}{a}=\frac{836{\cdot}4\times\sin 14^\circ.\,24'.\,25''}{212{\cdot}5};$$

$$\therefore\ L\sin B=\log 836{\cdot}4+L\sin 14^\circ.\,24'.\,25''-\log 212{\cdot}5$$
$$=2{\cdot}9224140+9{\cdot}3958630-2{\cdot}3273589$$
$$=9{\cdot}9909181;$$

$\therefore\ B=78^\circ.\,19'.\,24''$, or its supplement $101^\circ.\,40'.\,36''$.

(19) $$\sin B=\frac{b\,.\,\sin A}{a}=\frac{564{\cdot}8\times\sin 40^\circ.\,32'.\,16''}{379{\cdot}5};$$

$$\therefore\ L\sin B=\log 564{\cdot}8+L\sin 40^\circ.\,32'.\,16''-\log 379{\cdot}5$$
$$=2{\cdot}7518947+9{\cdot}8128794-2{\cdot}5792118$$
$$=9{\cdot}9855623;$$

$\therefore\ B=75^\circ.\,18'.\,28''{\cdot}2$, or its supplement $104^\circ.\,41'.\,31''{\cdot}8$.

(20) $$\sin B=\frac{b\,.\,\sin A}{a}=\frac{8032{\cdot}29\times\sin 71^\circ.\,3'.\,34''{\cdot}7}{9459{\cdot}31};$$

$$\therefore\ L\sin B=\log 8032{\cdot}29+L\sin 71^\circ.\,3'.\,34''{\cdot}7-\log 9459{\cdot}31$$
$$=3{\cdot}9048393+9{\cdot}9758256-3{\cdot}9758594$$
$$=9{\cdot}9048055;$$

$$\therefore\ B=53^\circ.\,26'.\,0''{\cdot}6.$$

EXAMPLES—LV. (p. 181).

(1) Let QP be the hill; $\angle QBP = 60°$; $\angle QAP = 45°$.

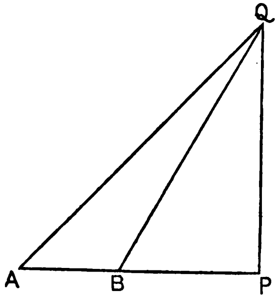


FIG. 30.

$$
\begin{aligned}
\text{Then } QP &= BP\,.\,\tan 60° \\
&= (AP - 100)\tan 60° \\
&= (QP - 100)\,.\,\sqrt{3}\,;
\end{aligned}
$$

$$\therefore QP = \frac{100\sqrt{3}}{\sqrt{3}-1} = \frac{100\sqrt{3}(\sqrt{3}+1)}{3-1} = 150 + 50\sqrt{3} = 236{\cdot}602 \ldots \text{ feet.}$$

(2) Let F be the fort; S_1 and S_2 the ships.

Then $\angle FS_1S_2 = 35°.\ 14'$, and $\angle FS_2S_1 = 42°.\ 12'$,

and $\angle S_1FS_2 = 180° - 77°.\ 26'$

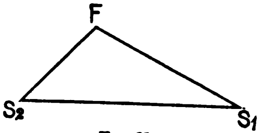


FIG. 31.

$$
\begin{aligned}
\text{and } FS_1 &= S_1S_2 \cdot \frac{\sin FS_2S_1}{\sin S_1FS_2} \\
&= 1760 \cdot \frac{\sin 42°.\ 12'}{\sin 77°.\ 26'} \\
&= 1760 \times \frac{{\cdot}671}{{\cdot}976} = 1210 \text{ yards,}
\end{aligned}
$$

$$\text{and } FS_2 = 1760 \times \frac{\sin 35°.\ 14'}{\sin 77°.\ 26'} = 1760 \times \frac{{\cdot}577}{{\cdot}976} = 1040{\cdot}5 \text{ yards.}$$

(3) With a construction similar to that in Example (2),

$$FS_1 = 880 \cdot \frac{\sin 85°.\ 15'}{\sin 11°} = 880 \times \frac{{\cdot}9965}{{\cdot}1908} = 4596 \text{ yards nearly,}$$

$$FS_2 = 880 \cdot \frac{\sin 83°.\ 45'}{\sin 11°} = 880 \times \frac{{\cdot}9940}{{\cdot}1908} = 4584{\cdot}48 \text{ yards.}$$

(4) Let AB be the flagstaff; BP the tower; Q the place of observation.

$$\text{Then } \tan BQA = \tan(AQP - BQP)$$

$$= \frac{\tan AQP - \tan BQP}{1 + \tan AQP \cdot \tan BQP}$$

$$= \frac{2{\cdot}05 - 2}{1 + 2{\cdot}05 \times 2} = \frac{{\cdot}05}{5{\cdot}1} = \frac{1}{102};$$

$$\therefore L \tan BQA = 10 + \log 1 - \log 102$$

$$= 10 - 2{\cdot}0086002$$

$$= 7{\cdot}9913998;$$

$$\therefore BQA = 33'.\ 42''.$$

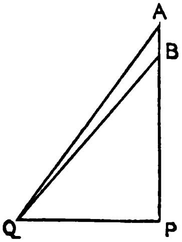


FIG. 32.

(5) Let A be the top of the steeple; D the top of the tower.

$$\angle APB = 60° \text{ and } \angle DPB = 45°.$$

$$\text{Then } BA = PB \cdot \tan 60°,$$

$$\text{and } BD = PB \cdot \tan 45°;$$

$$\therefore BA : BD = \tan 60° : \tan 45°$$

$$= \sqrt{3} : 1.$$

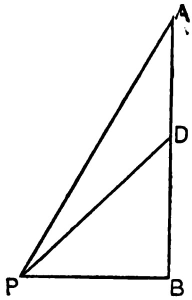


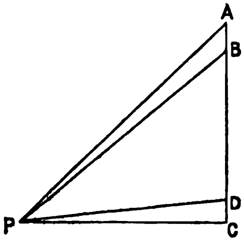
FIG. 33.

(6) Let PC be the river, CB the column, BA the statue. $CD = 6$ feet; and let x = breadth of river in feet.

$$\text{Then } \tan APB = \tan DPC = \frac{6}{x},$$

$$\tan APC = \frac{AC}{PC} = \frac{230}{x},$$

$$\tan BPC = \frac{200}{x}.$$



IG 34.

$$\text{Now } \tan BPC = \tan(APC - APB);$$

$$\therefore \frac{200}{x} = \frac{\frac{230}{x} - \frac{6}{x}}{1 + \frac{230}{x} \cdot \frac{6}{x}};$$

$$\therefore \frac{200}{x} = \frac{224x}{x^2 + 1380}, \text{ or } 24x^2 = 276000, \text{ or } x^2 = 11500;$$

$$\therefore x = 107{\cdot}2 \ldots \text{ feet.}$$

(7) Let A be the top of the pole ; B the top of the mound.

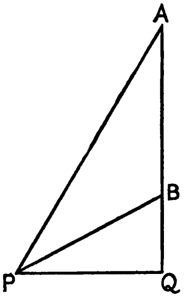


Fig. 85.

$$\angle APQ = 60^\circ; \quad \angle BPQ = 30^\circ.$$

$$\text{Then } AQ = PQ \,.\, \tan 60^\circ,$$

$$BQ = PQ \,.\, \tan 30^\circ;$$

$$\therefore AQ : BQ = \tan 60^\circ : \tan 30^\circ$$

$$= \sqrt{3} : \frac{1}{\sqrt{3}}$$

$$= 3 : 1;$$

$$\therefore AB = 2BQ.$$

(8) Let A be the top of the flagstaff ; B the top of the tower.

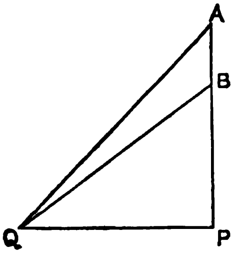


Fig. 86.

$$\text{Then } \angle BQP = 90^\circ - \angle AQP.$$

$$\text{Now } AB = AP - BP$$

$$= a(\tan AQP - \tan BQP)$$

$$= a.(\cot a - \tan a) = a \cdot \frac{\cos^2 a - \sin^2 a}{\cos a \,.\, \sin a}$$

$$= 2a \cdot \frac{\cos 2a}{\sin 2a} = 2a \,.\, \cot 2a.$$

(9) Let T be the place of the second observation.

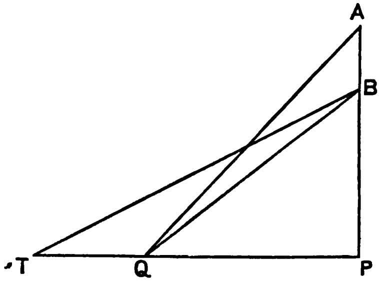


FIG. 37.

$$\text{Then } a = QP$$

$$= PT - TQ$$

$$= BP \,.\, \cot\frac{\alpha}{2} - c$$

$$= a\tan\alpha \,.\, \cot\frac{\alpha}{2} - c\,;$$

$$\therefore c = a\left(\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}\cot\frac{\alpha}{2} - 1\right) = a\left(\frac{2}{1-\tan^2\frac{\alpha}{2}} - 1\right) = a\cdot\frac{1+\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

$$= a\cdot\frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}} = \frac{a}{\cos\alpha}\,;$$

$\therefore a = c \,.\, \cos\alpha$, and putting this for a in the result of Example (8),

$$\text{length of flagstaff} = 2c \,.\, \cos\alpha \,.\, \cot 2\alpha = 2c \,.\, \cos\alpha\cdot\frac{\cos 2\alpha}{\sin 2\alpha}$$

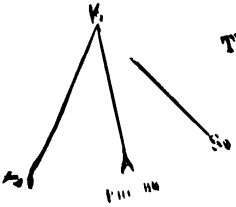
$$= 2c\cdot\frac{\cos 2\alpha}{2\sin\alpha} = c \,.\, \operatorname{cosec}\alpha \,.\, \cos 2\alpha.$$

[illegible] Let K be the kite; S_1 and S_2 the places of observation.

Draw KA perpendicular to S_1S_2.

Then, since the angles at S_1 and S_2 are equal,

KA bisects S_1S_2.



Then $KA = AS_1 . \tan KS_1A$

$= \frac{a}{2} . \tan\beta$

$= \frac{a}{2} . \sin\beta . \sec\beta$

$= \frac{a}{2} . \sin\alpha . \sec\beta$, because $\alpha = \beta$.

[illegible] Let AB be the smaller and PT the greater tower, and D the point midway between them.

Join TA, BP, DA, DP.

Then $\angle PDT = \angle DAB$.

Let $PT = x$ and $AB = y$.

[illegible] or $x = \frac{3600}{y}$.

Then [illegible]

Also $\tan PDT = \tan 2ATB$

$= \frac{2\tan ATB}{1-\tan^2 ATB}$,

But $\tan PDT = \frac{x}{120}$, and $\tan ATB = \frac{y}{120}$;

[illegible]

[illegible] 30 feet.

[illegible]

(12) Since $\angle ADB = \angle ACB$, a circle can be described about $ADCB$.

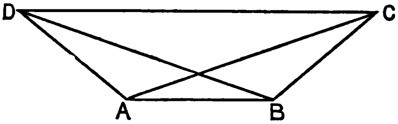


FIG. 40.

$$\therefore \angle ABD = \angle ACD = 19^\circ.\ 15',$$

$$\text{and } \angle DAC = 180^\circ - (40^\circ.\ 45' + 19^\circ.\ 15') = 120^\circ.$$

$$\therefore AB = \frac{AD \cdot \sin 30^\circ}{\sin 19^\circ.\ 15'},$$

$$\text{and } AD = \frac{DC \cdot \sin 19^\circ.\ 15'}{\sin 120^\circ};$$

$$\therefore \frac{AB}{DC} = \frac{\sin 30^\circ}{\sin 120^\circ} = \frac{\sin 30^\circ}{\sin 60^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

(13) Let x be the length of the zigzag road in miles.

$$\text{Then } 5 : 12 = \frac{5}{3} : x;$$

$$\therefore 5x = 20, \text{ or } x = 4 \text{ miles.}$$

(14) S_1 and S_2 are the two positions of the ship, A and B the two objects.

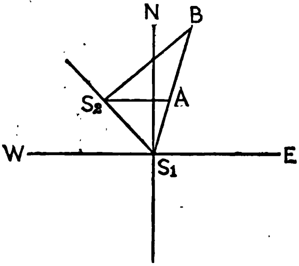


FIG. 41.

Then $\angle BS_1S_2 = 15^\circ + 45^\circ = 60^\circ$

$\angle BS_2S_1 = 90^\circ$(since N.W. is at right angles to N.E.)

$\angle S_2AS_1 = 180^\circ - (45^\circ + 60^\circ) = 75^\circ.$

$$\text{Then } BS_1 = S_1S_2 \,.\, \sec .\, BS_1S_2$$
$$= 5 \,.\, \sec 60° = 10,$$

$$\text{and } AS_1 = \frac{S_1S_2 \,.\, \sin AS_2S_1}{\sin S_2AS_1} = \frac{5 \,.\, \sin 45°}{\sin 75°} = \frac{5 \times \frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{10}{\sqrt{3}+1}.$$

$$\therefore AB = 10 - \frac{10}{\sqrt{3}+1} = \frac{10\sqrt{3}}{\sqrt{3}+1} = \frac{10\sqrt{3}(\sqrt{3}-1)}{3-1} = 5(3-\sqrt{3}).$$

(15) Let PQ be the tower.

Then AQP and PQB are right angles.

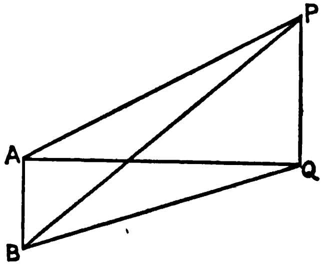


FIG. 42.

$\angle PAQ = 30°$, and $\angle PBQ = 18°$.

$$\text{Then } AQ = PQ \,.\, \cot 30° = PQ \times \sqrt{3},$$

$$BQ = PQ \,.\, \cot 18° = PQ \cdot \frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}.$$ (See Example XXXVI. 6.)

$$\text{Now } BQ^2 - AQ^2 = a^2;$$

$$\therefore PQ^2 \left\{ \frac{10+2\sqrt{5}}{6-2\sqrt{5}} - 3 \right\} = a^2;$$

$$\therefore PQ^2 \left\{ \frac{5+\sqrt{5}}{3-\sqrt{5}} - 3 \right\} = a^2;$$

$$\therefore PQ^2 \cdot \frac{4(\sqrt{5}-1)}{3-\sqrt{5}} = a^2;$$

$$\therefore PQ^2 \cdot \frac{4 \,.\, (2+2\sqrt{5})}{4} = a^2;$$

$$\therefore PQ = \frac{a}{\sqrt{(2+2\sqrt{5})}}.$$

(16) Let AB be the staff, C the centre of the ring in the vertical line ABC, D the extremity of the shadow; then if DE be drawn touching the ring in E, DE will be the direction of the sun, and CE is at right angles to DE.

Let $CE = r$, then $AB = AD = 8r$, and $AC = 9r$.

$$\therefore CD^2 = AC^2 + AD^2 = 145r^2,$$

$$\text{and } ED^2 = CD^2 - CE^2 = 144r^2;$$

$$\therefore ED = 12r.$$

$$\text{Hence } \tan ADC = \frac{9}{8}, \text{ and } \tan CDE = \frac{1}{12};$$

$$\therefore \tan ADE = \frac{\frac{9}{8} + \frac{1}{12}}{1 - \frac{9}{96}} = \frac{4}{3}.$$

$$\therefore \text{the sun's altitude} = \tan^{-1}\frac{4}{3}.$$

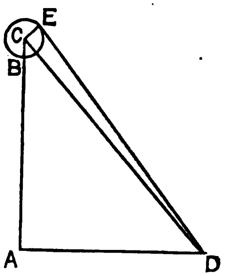


FIG. 43.

(17) Draw DM, EN perpendicular to CB, and let AB, BC, CA be represented by c, a, b.

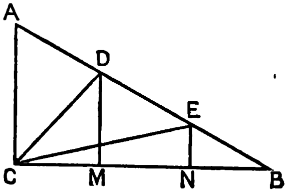


FIG. 44.

$$\text{Then } CD^2 = CM^2 + MD^2$$

$$= \frac{a^2}{9} + \frac{4b^2}{9} \qquad \text{(EUCLID, VI. 2, Ex. 1.)}$$

$$CE^2 = CN^2 + NE^2$$

$$= \frac{4a^2}{9} + \frac{b^2}{9}$$

$$DE^2 = \frac{c^2}{9};$$

$$\therefore CD^2 + CE^2 + DE^2 = \frac{5a^2}{9} + \frac{5b^2}{9} + \frac{a^2 + b^2}{9} = \frac{2}{3}(a^2 + b^2) = \frac{2}{3}c^2.$$

(18) Let P be the place of observation;

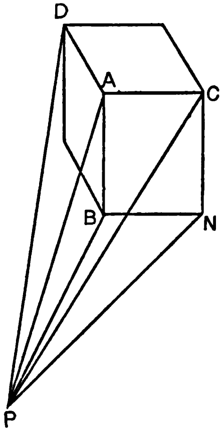


FIG. 45.

$$BN=x,\ PB=a.$$

Then $BA=a$, because $\angle APB=45°$,

$$PN=a\,.\,\cot 30°=a\sqrt{3},$$

$$\angle PBN=135°.$$

$$\text{Then } \cos PBN=\frac{PB^2+BN^2-PN^2}{2PB\,.\,BN},$$

$$\text{or } \cos 135°=\frac{a^2+x^2-3a^2}{2ax},$$

$$\therefore -\frac{1}{\sqrt{2}}=\frac{x^2-2a^2}{2ax},$$

$$\therefore \frac{1}{2}=\frac{x^4-4a^2x^2+4a^4}{4a^2x^2}.$$

Hence $x^4-6a^2x^2=-4a^4$, and $x^2-3a^2=\pm\sqrt{5}.a^2$,

and $\therefore\ x=a\sqrt{(3\pm\sqrt{5})}$.

(19) Let BA be the first tower; AC the moat; ED the other tower.

Draw EF parallel to DA. Let h=height of BA.

Then since $\angle BEA=\angle BCA=45°$, a circle can be described about $ABEC$, and since $\angle BAC=90°$, BC is the diameter of the circle, and therefore $\angle BEC=90°$.

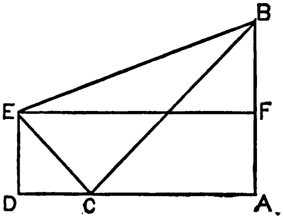


FIG. 46.

Then $CB^2=CA^2+BA^2=2h^2$,

and $CB^2=EC^2+EB^2$

$$=a^2+c^2+EF^2+FB^2$$

$$=a^2+c^2+(c+h)^2+(h-a)^2.$$

Hence

$$2h^2=a^2+c^2+c^2+2ch+h^2+h^2-2ah+a^2;$$

$$\therefore\ h=\frac{a^2+c^2}{a-c}.$$

(20) $$AC=AB\cdot\frac{\sin 15^\circ}{\sin 150^\circ}=100\cdot\frac{\sin 15^\circ}{\sin 30^\circ};$$

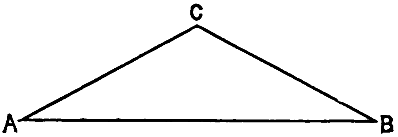


FIG. 47.

$$\therefore\ AC=100\times\frac{\sqrt{3}-1}{2\sqrt{2}}\div\frac{1}{2}=\frac{100(\sqrt{3}-1)}{\sqrt{2}}$$
$$=50(\sqrt{6}-\sqrt{2})=51{\cdot}76\ \ldots\ \text{feet.}$$

(21) Since BC points to N.W. the $\angle ABC=45^\circ$;

$\therefore\ \angle ACB=45^\circ$, and $AC=AB=10$ miles.

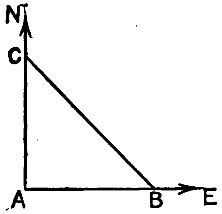


FIG. 48.

Also, $CB=\sqrt{AC^2+AB^2}=\sqrt{200}=14{\cdot}14\ \ldots$ miles.

(22) Let CA be a line from the end of the shadow in direction of the sun, AB the wall, BC the shadow.

$$\text{Then }\tan ACB=\frac{AB}{BC}=\frac{18}{16}=\frac{9}{8}.$$

$$\therefore\ \tan ACB=1{\cdot}125;$$

or, $ACB=\tan^{-1}1{\cdot}125$, which by the tables we find nearly equal to $48^\circ.22'$.

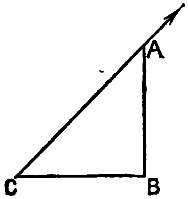


FIG. 49.

(23) Let AB be the spire; BP the tower; Q the place of observation.

Then $\angle BQP = 30°$, and $\angle AQP = 32°$.

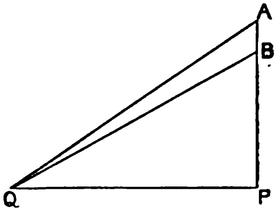


FIG. 50.

Now $AP = PQ \cdot \tan 32° = 200 \times {\cdot}6248694 = 124{\cdot}97398$

$BP = PQ \cdot \tan 30° = 200 \times {\cdot}5773503 = 115{\cdot}47006.$

$\therefore$ height of tower $= 115{\cdot}47$ yards nearly,

height of spire $= 9{\cdot}503$ yards nearly.

(24) $$\cos BAC = \frac{9 + 4 - \frac{324}{100}}{12} = \frac{61}{75} = {\cdot}8133333.$$

Hence, by the tables, $\angle BAC = 35°.\,34'.\,32''$,

and $\therefore \angle BAD = 144°.\,25'.\,28''$.

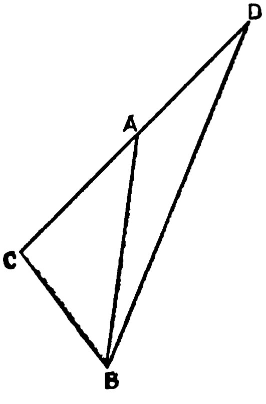


FIG. 51.

Next, $BD=\dfrac{AB\,.\,\sin 144^\circ.\,25'.\,28''}{\sin 17^\circ.\,47'.\,20''}$

$=\dfrac{3\times{\cdot}5817759}{{\cdot}3055106}=5{\cdot}71307\ .\ .\ .$ miles.

(25) $\angle BAC=17^\circ.\,44'$,

$AB=BC\cdot\dfrac{\sin 139^\circ.\,58'}{\sin 17^\circ.\,44'}$.

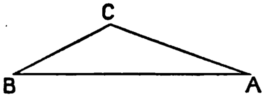


Fig. 52.

Hence, by the tables,

$AB=\dfrac{840{\cdot}5\times{\cdot}6432332}{{\cdot}3045872}$ yards $=1775$ yards nearly;

$\therefore$ AB differs from a mile by about 15 yards.

(26) $\angle BCA=180^\circ-(50^\circ.\,20'+110^\circ.\,12')=19^\circ.\,28'$;

$\therefore\ BC=AB\cdot\dfrac{\sin 50^\circ.\,20'}{\sin 19^\circ.\,28'}$.

$\therefore\ \log BC=\log 2700+L\sin 50^\circ.\,20'-L\sin 19^\circ.\,28'$

$=3{\cdot}4313638+9{\cdot}8863616-9{\cdot}5227811$

$=3{\cdot}7949443$.

Hence $BC=6236{\cdot}549$ feet.

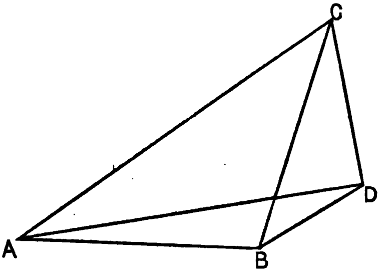


Fig. 53.

Next, if CD be the height of the mountain,

$CD=BC\sin CBD$,

$=6236{\cdot}549\times\sin 10^\circ.\,7'$

$=6236{\cdot}549\times{\cdot}1756531$

$=1095{\cdot}47\ .\ .\ .$ feet.

(27)

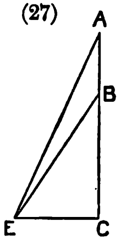


FIG. 54.

Let $EC = x$ feet.

Then $\tan AEB = \tan(AEC - BEC)$;

$$\therefore \tan 10^\circ = \frac{\tan AEC - \tan BEC}{1 + \tan AEC . \tan BEC}.$$

$$\therefore \cdot 176327 = \frac{\frac{60}{x} - \frac{40}{x}}{1 + \frac{2400}{x^2}}.$$

$$\therefore \cdot 176327 = \frac{20x}{x^2 + 2400};$$

and solving this quadratic we get

$$x = 85 \cdot 28, \text{ or } 28 \cdot 14.$$

(28) Let CP be the height of the hill.

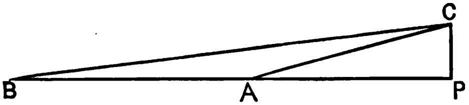


FIG. 55.

$$\text{Then } CA = AB . \frac{\sin ABC}{\sin ACB}$$

$$= 1760 \times \frac{\sin 2^\circ . 45'}{\sin 9^\circ . 28'}$$

$$= \frac{1760 \times \cdot 0479781}{\cdot 1644738}$$

$$= 513 \cdot 4 \text{ nearly};$$

$$\therefore CP = 513 \cdot 4 \times \sin CAP$$

$$= 513 \cdot 4 \times \cdot 2116091 = 108 \cdot 64 \ldots \text{ yards.}$$

(29) Let AB be the tower, S the ship.

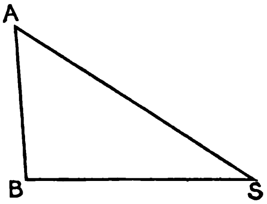


FIG. 56.

$$\begin{aligned}\text{Then } BS &= AB \,.\, \cot ASB\\ &= 150 \times 1{\cdot}3613350\\ &= 204{\cdot}2 \;.\;.\;.\; \text{feet.}\end{aligned}$$

$$(30)\quad \tan\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \cot\frac{C}{2}$$

$$= \frac{3225{\cdot}77}{9541{\cdot}29}\cot 18^\circ.\,43'.$$

$$L\tan\frac{A-B}{2} = 3{\cdot}5086333 - 3{\cdot}9795979 + 10{\cdot}4700495$$

$$= 9{\cdot}9990849.$$

$$\text{Hence } \frac{A-B}{2} = 44^\circ.\,56'.\,20'',$$

$$\text{and } \frac{A+B}{2} = 71^\circ.\,17'\,;$$

$$\therefore\; A = 116^\circ.\,13'.\,20'', \text{ and } B = 26^\circ.\,20'.\,40''.$$

$$\text{Also } c = \frac{b\,.\,\sin C}{\sin B} = 3157{\cdot}76 \times \frac{{\cdot}6078379}{{\cdot}4437665} = 4325{\cdot}26.$$

(31) $\angle\, BS_1S_2 = 55^\circ$, and $\angle\, BS_2S_1 = 62^\circ.\,30'$;

$\therefore\; \angle\, S_1BS_2 = 180^\circ - (55^\circ + 62^\circ.30') = 62^\circ.30'$;

$\therefore\; S_1S_2 = BS_1 = 1$ mile.

$$\text{Then } S_2B = \frac{S_1B\,.\,\sin BS_1S_2}{\sin BS_2S_1}$$

$$= \frac{1 \times \sin 55^\circ}{\sin 62^\circ.30'} = \frac{{\cdot}8191520}{{\cdot}8870108}$$

$$= {\cdot}923497 \text{ miles.}$$

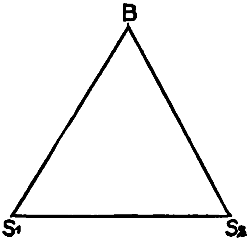


FIG 57.

(32) From E, the lower window, draw EB perpendicular to the tower AB; from D, the upper window, draw DC perpendicular to the tower.

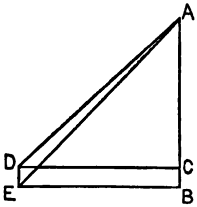


FIG. 58.

Then $\angle AEB = 45°$,

and $\angle ADC = 40°$,

and $DC = EB = AB$.

$$\therefore DC = 20 + AC$$
$$= 20 + DC \,.\, \tan 40°.$$
$$\therefore DC = \frac{20}{1 - \tan 40°} = \frac{20}{1 - \cdot 8390996}$$
$$= \frac{20}{\cdot 1609004} = 124 \cdot 3 \ldots \text{ feet.}$$

(33)

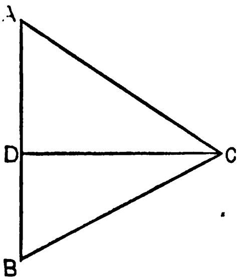


FIG. 59.

Let CD be the perpendicular breadth of the river.

Now $\angle ACB = 180° - (50° + 65°) = 65°$.

$$\therefore AC = AB = 400 \text{ yards.}$$

Hence $CD = AC \,.\, \sin 50°$

$$= 400 \times \cdot 7660444 = 306 \cdot 4178 \text{ yards.}$$

(34) Let AB, AC be the lines of the railways, D the point at which the train travelling 30 miles an hour is in $2\frac{1}{2}$ hours.

The other train may then be at M or N, points on AB equidistant from D, and such that $MD = DN = 50$ miles.

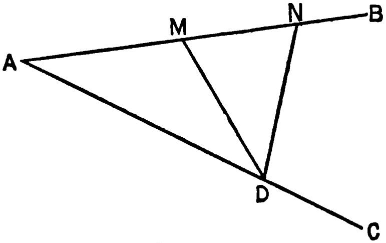


FIG. 60

Also, $AD=75$ miles.

$$\text{Then } \sin AND=\frac{75\,.\,\sin 35^\circ.\,20'}{50}=\frac{3}{2}\times\cdot5783323=\cdot8674984.$$

$$\text{Hence } \angle AND=60^\circ.\,10' \text{ nearly;}$$

$$\therefore \angle ADN=84^\circ.\,30',$$

$$\text{and } AN=\frac{50\times\sin 84^\circ.\,30'}{\sin 35^\circ.\,20'}=\frac{50\times\cdot9953962}{\cdot5783323}=\frac{49\cdot7698100}{\cdot5783323}\text{ miles.}$$

$$\therefore \text{rate of train}=\frac{49\cdot7698100}{\cdot5783323}\div2\tfrac{1}{2}=34\cdot42284\ .\ .\ .\text{ miles per hour.}$$

$$\text{Next, } \angle DMN=\angle AND=60^\circ.\,10' \text{ nearly;}$$

$$\therefore \angle AMD=119^\circ.\,50' \text{ nearly;}$$

$$\therefore \angle ADM=24^\circ.\,50' \text{ nearly,}$$

$$\text{and } AM=\frac{AD\,.\,\sin ADM}{\sin AMD}=\frac{75\,.\,\sin 24^\circ.\,50'}{\sin 119^\circ.\,50'}=75\times\frac{\cdot4199801}{\cdot8674984}\text{ miles;}$$

$$\therefore \text{rate of train}=75\times\frac{\cdot4199801}{\cdot8674984}\div2\tfrac{1}{2}=14\cdot524\ .\ .\ .\text{ miles per hour.}$$

(35) Let AB be the base of 600 yards ; C the tree ; CD a perpendicular on AB.

$$\text{Then } \angle ACB=180^\circ-(52^\circ.\,14'+68^\circ.\,32')$$
$$=59^\circ.\,14'.$$

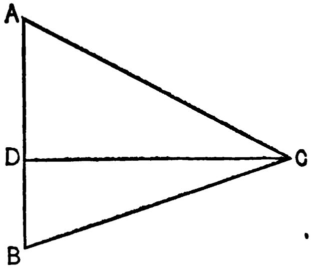


FIG. 61.

$$\text{Now } CD=AC\,.\,\sin CAD$$
$$=\frac{600\,.\,\sin ABC}{\sin ACB}\,.\,\sin CAD$$
$$=\frac{600\,.\,\sin 68^\circ.\,32'.\,\sin 52^\circ.\,14'}{\sin 59^\circ.\,14'}$$
$$=\frac{600\times\cdot9306306\times\cdot7905115}{\cdot8592576}=513\cdot7045\text{ yards.}$$

(36) Let AB be the tower ; C the first place of observation ; D the second place of observation.

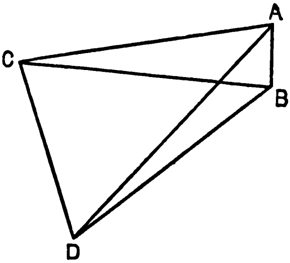


Fig. 62.

Then ACD and ABD are right angles.

$$\text{Now } AC=AB\,.\,\text{cosec}ACB$$
$$=100\times\text{cosec}50^\circ=130{\cdot}54073.$$
$$AD=\sqrt{(300)^2+(130{\cdot}54073)^2}$$
$$=\sqrt{107040{\cdot}127569}$$
$$=327{\cdot}16\ .\ .\ .$$
$$\sin\angle ADB=\frac{AB}{AD}=\frac{100}{327{\cdot}16}={\cdot}3056608.$$

Hence $\angle ADB=17^\circ.\ 47'.\ 50''$.

(37)

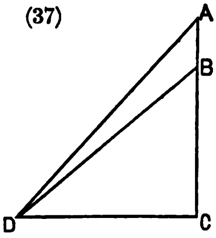


Fig. 63.

Let $x=$height of tower in yards ;

$$\tan 48'.\ 20''=\tan(ADC-BDC)$$
$$=\frac{\dfrac{x+4}{100}-\dfrac{x}{100}}{1+\dfrac{x\,.\,(x+4)}{10000}};$$
$$\therefore\ {\cdot}0140605=\frac{400}{10000+x^2+4x}.$$

Solving this quadratic we get $x=134$ yards nearly.

(38) Let A be the object ; AB a vertical line meeting the horizontal plane through C in B ; D the point 300 yards up the hill.

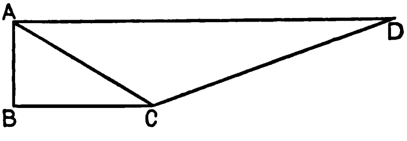


Fig. 64.

Then $\angle BCA=29^\circ.\ 12'.\ 40''=\angle CAD$.

$\angle CDA=16^\circ$.

$$\text{Then } CA=\frac{CD\,.\,\sin 16^\circ}{\sin 29^\circ.\ 12'.\ 40''}=\frac{300\times{\cdot}2756374}{{\cdot}4880290}=169{\cdot}4392\text{ yards.}$$

(39) At the end of three hours each engine has passed over 90 miles. Let AB, AC be the distances traversed.

Draw AD perpendicular to BC.

Then $\angle BAD = 25^\circ.\ 10'$,

and $BD = AB \,.\, \sin BAD$

$= 90 \times \cdot 4252528.$

$\therefore\ BC = 2 \times 90 \times \cdot 4252528 = 76\cdot 5455\ldots$ miles.

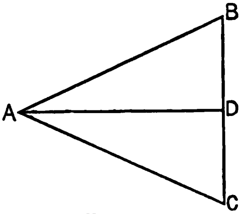


FIG. 65.

(40) Diagram as in Example (37); and x = height of tower in feet.

$$\tan ADB = \frac{6}{150} = \frac{1}{25};$$

$$\therefore\ \frac{1}{25} = \frac{\dfrac{x+30}{150} - \dfrac{x}{150}}{1 + \dfrac{x(x+30)}{22500}} = \frac{4500}{x^2 + 30x + 22500.}$$

Solving this quadratic $x = 285$ feet nearly.

EXAMPLES—LVI. (p. 199).

1. Area $= \frac{1}{2}\left\{10 \times 12 \times \sin 60^\circ\right\}$ square inches

$= \left(60 \times \frac{\sqrt{3}}{2}\right)$ square inches $= 30\sqrt{3}$ square inches.

2. Area $= \frac{1}{2}\left\{40 \times 60 \times \sin 30^\circ\right\}$ square feet

$= \left(1200 \times \frac{1}{2}\right)$ square feet $=$ 600 square feet.

3. Area $= \frac{1}{2}\left\{4 \times 3\frac{3}{4}\right\}$ square feet $= 7\frac{1}{2}$ square feet.

4. Area $= \sqrt{s\,.\,(s-a)\,.\,(s-b)\,.\,(s-c)} = \sqrt{8 \times 3 \times 2 \times 3} = 4 \times 3$

$= 12$ square inches.

5. Area $= \sqrt{1017 \times 392 \times 512 \times 113} = \sqrt{9 \times 113 \times 8 \times 49 \times 8 \times 64 \times 113}$

$= (3 \times 113 \times 8 \times 7 \times 8) = 151872.$

6. Area $=\sqrt{544\times135\times375\times34}=\sqrt{17\times32\times15\times9\times125\times3\times17\times2}$

$$=17\times8\times9\times25=30600.$$

7. Area $=\sqrt{585\times8\times512\times65}=\sqrt{5\times13\times9\times8\times64\times8\times13\times5}$

$$=5\times13\times3\times8\times8=12480.$$

8.

$$\begin{aligned}s.(s-c)&=\frac{a+b+c}{2}\cdot\frac{a+b-c}{2}\\&=\frac{(a+b)^2-c^2}{4}\\&=\frac{(a+b)^2-(a^2+b^2)}{4}\\&=\frac{2ab}{4}\\&=\frac{ab}{2}=\text{area of the triangle.}\end{aligned}$$

9. Area $=\sqrt{\frac{146{\cdot}27}{2}\times\frac{41{\cdot}21}{2}\times\frac{48{\cdot}75}{2}\times\frac{56{\cdot}31}{2}}$

$$\begin{aligned}\therefore\ \log\text{area}&=\frac{1}{2}\left\{\log146{\cdot}27+\log41{\cdot}21+\log48{\cdot}75+\log56{\cdot}31-4\log2\right\}\\&=\frac{1}{2}\left\{2{\cdot}1651553+1{\cdot}6150026+1{\cdot}6879746\right.\\&\qquad\left.+1{\cdot}7505855-1{\cdot}2041200\right\}\\&=3{\cdot}0072990;\end{aligned}$$

$$\therefore\ \text{area}=1016{\cdot}9487.$$

10. Let a, b, c be in descending arithmetical progression;

$$\text{then } a+c=2b.$$

Thus the perimeter is $3b$, and the side of an equilateral triangle of equal perimeter is b.

$$\text{Then }\sqrt{s.(s-a)(s-b)(s-c)}=\frac{3}{5}\cdot\frac{1}{2}\cdot b^2.\sin60^\circ,$$

$$\text{or }\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}=\frac{3\sqrt{3}}{20}b^2,$$

$$\text{or}\ \sqrt{3b^2(b+c-a)(a+b-c)}=\frac{3\sqrt{3}}{5}b^2$$

$$\sqrt{(b+c-a)(a+b-c)}=\frac{3}{5}b$$

$$\sqrt{\frac{3c-a}{2}\cdot\frac{3a-c}{2}}=\frac{3}{10}(a+c)$$

$$\frac{10ac-3a^2-3c^2}{4}=\frac{9}{100}(a^2+c^2).$$

Solving this quadratic we get $\frac{a}{c}=\frac{7}{3}$ or $\frac{3}{7}$.

Hence the sides are proportional to 7, 5, 3.

Then $\cos A=\frac{b^2+c^2-a^2}{2bc}=-\frac{1}{2}$;

and $\therefore A=120°$.

11. Let AEB be the triangular part turned down.

Then area of $AEB=\frac{1}{2}AB.AE.$

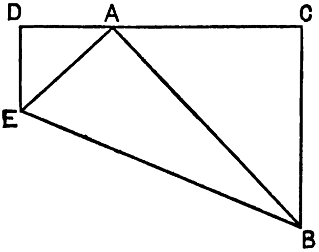


FIG. 66.

But $\frac{AE}{AD}=\frac{AB}{BC}$, by similar triangles AED, BAC;

$$\therefore \text{area of } AEB=\frac{1}{2}AB\cdot\frac{AB.AD}{BC}$$

$$=\frac{1}{2}\cdot\frac{AB^2}{BC}\cdot(CD-AC)$$

$$=\frac{1}{2}\cdot\frac{AB^2}{BC}\cdot\left\{AB-\sqrt{(AB^2-BC^2)}\right\}$$

12. $$\text{Area}=\frac{bc\,.\,\sin A}{2}=\frac{b\sin A\,.\,c\sin A}{2\sin A}=\frac{a\sin B\,.\,a\sin C}{2\sin A}=\frac{a^2\sin B\,.\,\sin C}{2\sin(B+C)}.$$

13. $$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\left(\frac{a^2}{\sin A}+\frac{b^2}{\sin B}+\frac{c^2}{\sin C}\right)$$

$$=\sqrt{\frac{(s-b)(s-c)}{bc}}\,.\,\sqrt{\frac{(s-a)(s-c)}{ac}}\,.\,\sqrt{\frac{(s-a)(s-b)}{ab}}.\left(\frac{a^2bc}{2S}+\frac{b^2ac}{2S}+\frac{c^2ba}{2S}\right)$$

$$=\frac{(s-a)(s-b)(s-c)}{abc}\,.\,\frac{abc(a+b+c)}{2S}$$

$$=\frac{s.(s-a)(s-b)(s-c)}{S}=S.$$

14. $$R=\frac{abc}{4S}\text{ and }r=\frac{S}{s};$$

$$\therefore\ \frac{abc}{4S}=\frac{2S}{s};$$

$$\therefore\ abc=\frac{8S^2}{s}$$

$$=8(s-a)(s-b)(s-c)$$

$$=(b+c-a)(a+c-b)(a+b-c).$$

Squaring both sides,—

$$a^2b^2c^2=\{a+(b-c)\}\{a-(b-c)\}\times\{b+(a-c)\}\{b-(a-c)\}$$
$$\times\{c+(a-b)\}\{c-(a-b)\}$$

$$=\{a^2-(b-c)^2\}\{b^2-(a-c)^2\}\{c^2-(a-b)^2\}.$$

Now this equality can only exist when $a=b=c$, for in any other case each factor on the right-hand side is less than the corresponding factor on the left-hand side.

15. $$\frac{b-c}{a}=\frac{\sin B-\sin C}{\sin A}=\frac{2\cos\frac{B+C}{2}\cdot\sin\frac{B-C}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}=\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}};$$

$$\therefore\ (b-c)\cos\frac{A}{2}=a\,.\,\sin\frac{B-C}{2}.$$

16. OA bisects $\angle A$, and FE at right angles;

$$\begin{aligned}\therefore \text{ area } FOE &= FH\,.\,OH\\ &= r\cos\frac{A}{2}\cdot r\sin\frac{A}{2}\\ &= r^2\cdot\frac{1}{2}\sin A\\ &= \frac{S^2}{s^2}\cdot\frac{S}{bc}\\ &= \frac{S^3}{s^2\,.\,bc}.\end{aligned}$$

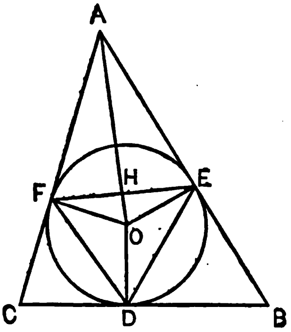


Fig. 67.

$\therefore$, by symmetry,

$$\begin{aligned}\text{area } FDE &= \frac{S^3}{s^2}\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)\\ &= \frac{S^3\,.\,2s}{s^2.abc}\\ &= \frac{2}{abc}\cdot\frac{\{s\,.\,(s-a)(s-b)(s-c)\}^{\frac{3}{2}}}{s}\\ &= \frac{2}{abc}\cdot s^{\frac{1}{2}}\left\{(s-a)(s-b)(s-c)\right\}^{\frac{3}{2}}.\end{aligned}$$

17. Area of quadrilateral

$$=\frac{1}{2}\left\{EA\,.\,AD+DA\,.\,AC+BA\,.\,AC+BA\,.\,AE\right\}\sin A$$

$$=\frac{1}{2}\left\{(EA+AC)\,.\,AD+(EA+AC)BA\right\}\sin A$$

$$=\frac{1}{2}\cdot EC\,.\,BD\,.\sin A$$

$$=\frac{1}{2}ab\,.\sin A.$$

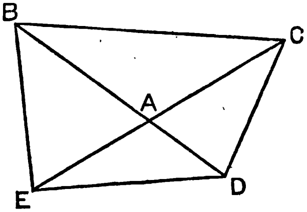


Fig. 68.

18. $$\frac{a^2-b^2}{2}\cdot\frac{\sin A\,.\,\sin B}{\sin(A-B)}=\frac{a^2\sin A\,.\,\sin B-b^2\sin A\,.\,\sin B}{2\sin(A-B)}$$

$$=\frac{ab\sin^2 A-ab\sin^2 B}{2\sin(A-B)}=\frac{ab.\sin(A+B).\sin(A-B)}{2\sin(A-B)}$$

$$=\frac{ab\sin(A+B)}{2}$$

$$=\frac{ab\,.\sin C}{2}=\text{area of triangle.}$$

19. $$R=\frac{a}{2\sin A}=\frac{a}{\sqrt{2}}$$

$$r^a=\frac{S}{s-a}=\frac{\frac{1}{2}ab}{\frac{2a+c}{2}-a}=\frac{ab}{c}=\frac{a^2}{a\sqrt{2}}=\frac{a}{\sqrt{2}};$$

$$\therefore\ R=r_a.$$

20. $$\cot(B-A)+\cot 2\left(A+\frac{C}{2}\right)=\cot(B-A)+\cot(2A+C)$$
$$=\frac{1+\cot B\,.\cot A}{\cot B-\cot A}+\frac{1-\cot 2A\,.\cot C}{\cot 2A+\cot C}$$
$$=\frac{1+1}{\tan A-\cot A}+\frac{1-0}{\cot 2A+0}$$
$$=\frac{2}{\tan A-\cot A}+\frac{2\tan A}{1-\tan^2 A}$$
$$=\frac{2\tan A}{\tan^2 A-1}+\frac{2\tan A}{1-\tan^2 A}$$
$$=0.$$

21. $$\frac{2abc}{a+b+c}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$
$$=\frac{2abc}{a+b+c}\cdot\sqrt{\frac{s\,.(s-a)}{bc}}\cdot\sqrt{\frac{s\,.(s-b)}{ac}}\cdot\sqrt{\frac{s\,.(s-c)}{ab}}$$
$$=\frac{2abc}{a+b+c}\cdot\frac{s}{abc}\cdot\sqrt{s\,.(s-a)(s-b)(s-c)}$$
$$=\sqrt{s\,.(s-a)(s-b)(s-c)}$$
=area of triangle.

22. $$\frac{\sin 2A(2a+c)^2}{32\,.\cos^4\frac{A}{2}}=\frac{\sin 2A\,.(2s)^2}{32\,.\frac{s^2\,.(s-a)^2}{b^2c^2}}$$
$$=\frac{\sin 2A\,.b^2c^2}{8\,.(s-a)^2}$$
$$=\frac{\sin 2A\,.b^2c^2}{8\left(\frac{c}{2}\right)^2}$$

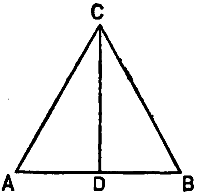


FIG. 69.

$$=\frac{b^2\,.\sin 2A}{2}=b^2\,.\sin A\,.\cos A=b^2\,.\sin A\,.\frac{c}{2b}$$
$$=\frac{1}{2}bc\,.\sin A$$
=area of triangle.

$$\therefore\ \text{area}\times 32\cos^4\frac{A}{2}=\sin 2A\,.(2a+c)^2.$$

23.
$$AD=b\,.\,\sin C, \;\therefore,\; AD\,.\,b=b^2.\sin C.$$
$$AD=c\,.\,\sin B, \;\therefore,\; AD\,.\,c=c^2.\sin B.$$

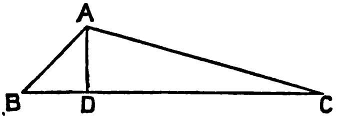


FIG. 70.

$$\therefore\; AD(b+c)=b^2.\sin C+c^2.\sin B;$$
$$\therefore\; AD=\frac{b^2\sin C+c^2\sin B}{b+c}.$$

24. (1)
$$BD=r\,.\cot\frac{B}{2}$$
$$CD=r\,.\cot\frac{C}{2};$$

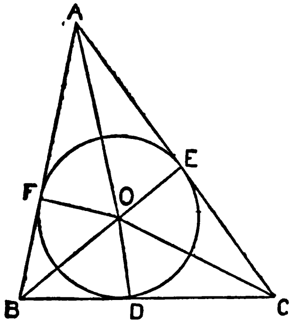


FIG. 71.

$$\therefore\; r\,.\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=BD+CD=a.$$
$$\therefore\; r=\frac{a}{\cot\frac{B}{2}+\cot\frac{C}{2}}.$$

(2) From the preceding Example—

$$r=\frac{a\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\sin\left(\frac{B+C}{2}\right)}$$

$$=\frac{2R\,.\,\sin A\,.\,\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{A}{2}}\text{, by Art. 221.}$$

(3) Let O be the centre of the escribed circle touching BC and the other sides produced, as in diagram to Art. 223.

Then $BD=OD\,.\,\cot OBD=r_1\,.\,\tan\frac{B}{2}$,

and $CD=OD\,.\,\cot OCD=r_1\,.\,\tan\frac{C}{2}$.

$\therefore\ BD+CD=r_1\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$;

$$\therefore\ r_1=\frac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}.$$

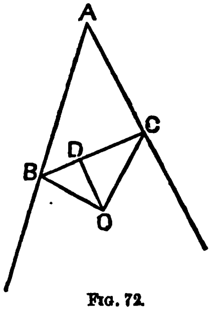


FIG. 72.

(4) By the preceding Example—

$$r_1=\frac{a\,.\,\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\sin\frac{B+C}{2}}$$

$$=\frac{2R\,.\,\sin A\,.\,\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=4R\,.\,\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}.$$

(5)
$$r_1=4R\,.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2},$$

$$r_2=4R\,.\cos\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2},$$

$$r_3=4R\,.\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2};$$

$$\therefore\; r_1+r_2+r_3=4R\,.\cos\frac{A}{2}\cdot\left(\sin\frac{B}{2}\cdot\cos\frac{C}{2}+\cos\frac{B}{2}\cdot\sin\frac{C}{2}\right)$$
$$+4R\,.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$=4R\,.\cos\frac{A}{2}\cdot\cos\frac{A}{2}+4R\,.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$=2R\,.(\cos A+1)+R\,.(\cos B+\cos C-\cos A+1)$$

Ex. xlviii. 12.

$$=3R+R(\cos A+\cos B+\cos C).$$

(6)
$$R+r=R+\frac{2R\sin A\,.\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{A}{2}}\text{ by (2)}$$

$$=R+4R\,.\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$$

$$=R\,.\left(1+4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)$$

$$=R(\cos A+\cos B+\cos C).$$

Ex. xlviii. 8.

25. Let r be the radius of the circle.

Then area of inscribed polygon of $2n$ sides $=nr^2.\sin\frac{\pi}{n}$;

area of inscribed polygon of n sides $=\frac{nr^2}{2}\cdot\sin\frac{2\pi}{n}$;

area of circumscribed polygon of n sides $=nr^2.\tan\frac{\pi}{n}$.

$$\text{And}\left(\frac{nr^2}{2}\cdot\sin\frac{2\pi}{n}\right)\times\left(nr^2.\tan\frac{\pi}{n}\right)$$

$$=\frac{n^2.r^4}{2}\cdot 2\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}\cdot\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}$$

$$=n^2r^4.\sin^2\frac{\pi}{n}$$

$$=\left(nr^2.\sin\frac{\pi}{n}\right)^2.$$

26. Let O, M be the centres of the inscribed and escribed circles.

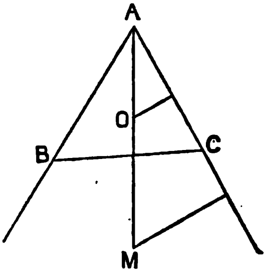


FIG. 73.

Then $MO=MA-OA$

$$=r_1\text{cosec}\frac{A}{2}-r.\text{cosec}\frac{A}{2}$$

$$=(r_1-r)\text{cosec}\frac{A}{2}$$

$$=\left\{4R.\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}-4R.\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right\}\text{cosec}\frac{A}{2}$$

(By Ex. 24.)

$$=4R\left\{\cos\frac{B}{2}\cdot\cos\frac{C}{2}-\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right\}$$

$$=4R.\sin\frac{A}{2},$$

and similarly for the other escribed circles.

(27) Let *DEF* be the triangle so formed.

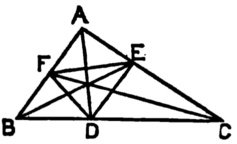


Fig. 74.

Then since $\frac{BD}{CD}=\frac{c}{b}$,

$$\frac{BD}{BC}=\frac{c}{b+c}, \text{ or, } BD=\frac{ac}{b+c}.$$

So also, $CD=\frac{ab}{b+c}$, and similarly for the segments of the other sides.

Then area $CDE=\frac{1}{2}\cdot\frac{ab}{b+c}\cdot\frac{ab}{a+c}\cdot\sin C=\frac{S.ab}{(a+c)(b+c)}$.

Similar expressions may be obtained for the areas of *BFD*, *AFE*.

$$\therefore \text{ area of } DEF=S\left\{1-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(b+a)(c+a)}-\frac{ca}{(c+b)(a+b)}\right\}$$

$$=\frac{2abc.S}{(a+b)(b+c)(c+a)}=2S.\frac{a}{b+c}\cdot\frac{b}{c+a}\cdot\frac{c}{a+b}.$$

$$\text{Now, } \frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}$$

$$\frac{b}{c+a}=\frac{\sin\frac{B}{2}}{\cos\frac{C-A}{2}}, \text{ and } \frac{c}{a+b}=\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}$$

$$\therefore \frac{\text{area} DEF}{\text{area} ABC}=\frac{2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{B-C}{2}\cdot\cos\frac{C-A}{2}\cdot\cos\frac{A-B}{2}}.$$

28. $r_1r_2+r_2r_3+r_3r_1=\frac{S^2}{(s-a)(s-b)}+\frac{S^2}{(s-b)(s-c)}+\frac{S^2}{(s-c)(s-a)}$

$$=s.(s-c)+s.(s-a)+s.(s-b)$$

$$=s.\{3s-(a+b+c)\}$$

$$=s^2.$$

29. $$\frac{\sin BAD}{\sin ADB}=\frac{BD}{AB}.$$

$$\frac{\sin ABC}{\sin ACB}=\frac{AC}{AB}.$$

$\therefore$ since $\sin ADB=\sin ACB$,

$$\frac{\sin BAD}{\sin ABC}=\frac{BD}{AC};$$

$$\therefore AC\sin A=BD.\sin B.$$

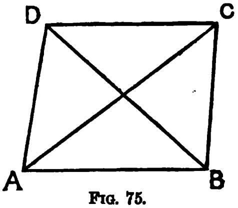


FIG. 75.

30. Let O, P be the centres of the inscribed and one of the escribed circles.

Then OB and PB bisect the interior and exterior angles at B; and $\therefore$ OBP is a right angle.

Hence $OBPC$ is a quadrilateral round which a circle may be described.

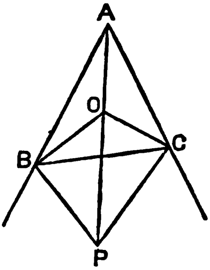


FIG. 76.

$$\begin{aligned}\text{Then } OP&=OB.\sec BOP\\&=OB.\sec BCP\\&=OB.\operatorname{cosec}\frac{C}{2}.\end{aligned}$$

$$\text{And } OB=\frac{c.\sin\frac{A}{2}}{\sin AOB}=\frac{c.\sin\frac{A}{2}}{\cos\frac{C}{2}};$$

$$\therefore OP=\frac{c.\sin\frac{A}{2}}{\sin\frac{C}{2}\cdot\cos\frac{C}{2}}=\frac{2c.\sin\frac{A}{2}}{\sin C}=\frac{2a.\sin\frac{A}{2}}{\sin A}=\frac{a}{\cos\frac{A}{2}}.$$

$$\text{Similarly } OP=\frac{b}{\cos\frac{B}{2}}=\frac{c}{\cos\frac{C}{2}}.$$

31. $$r.\cos\frac{A}{2}\cdot\operatorname{cosec}\frac{B}{2}\cdot\operatorname{cosec}\frac{C}{2}=\frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\cdot\sin\frac{C}{2}},$$

$$=r\cdot\frac{\sqrt{\frac{s.(s-a)}{bc}}}{\sqrt{\frac{(s-a).(s-c)}{ac}}\cdot\sqrt{\frac{(s-b)(s-a)}{ab}}}$$

$$=r\cdot\frac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$=\frac{S}{s}\cdot\frac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$=a.$$

32. $$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}$$

$$=\sqrt{\frac{(s-c)(s-b)}{s.(s-a)}}+\sqrt{\frac{(s-a)(s-c)}{s.(s-b)}}+\sqrt{\frac{(s-b)(s-a)}{s.(s-c)}}$$

$$=\frac{(s-c)(s-b)}{S}+\frac{(s-a)(s-c)}{S}+\frac{(s-b)(s-a)}{S}$$

$$=\frac{1}{4S}\cdot\left\{(a+b-c).(a+c-b)+(b+c-a).(a+b-c)\right.$$
$$\left.+(a+c-b).(b+c-a)\right\}$$

$$=\frac{1}{4S}\left\{2ab+2ac+2bc-a^2-b^2-c^2\right\}$$

$$=\frac{1}{S}\cdot\left\{ab+ac+bc-\frac{a^2+b^2+c^2+2ab+2ac+2bc}{4}\right\}$$

$$=\frac{1}{S}\left\{ab+ac+bc-s^2\right\}$$

$$=\frac{ab+ac+bc}{S}-\frac{s^2}{S}$$

$$=\frac{4R}{abc}\cdot(ab+ac+bc)-\frac{s}{r}$$

$$=4R.\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{s}{r}.$$

33. $PA \,.\, BC = BA \,.\, PC + AC \,.\, BP$ (EUCLID, VI. D.)

$$\text{and } \frac{\sin A}{BC} = \frac{\sin C}{BA} = \frac{\sin B}{AC}.$$

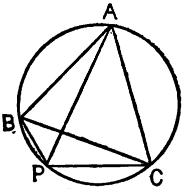


FIG. 77.

$$\therefore PA \,.\, \sin A = PC \,.\, \sin C + PB \,.\, \sin B.$$

34. Each of the angles at $O = 120°$.

Let OA, OB, OC be represented by d_1, d_2, d_3.

$c^2 = d_1^2 + d_2^2 - 2d_1d_2 \,.\, \cos 120°$;

$$\therefore c = \sqrt{d_1^2 + d_2^2 + d_1d_2}.$$

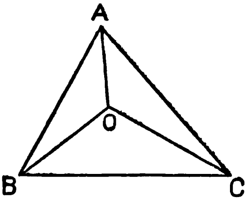


FIG. 78.

Similarly for a and b.

Also, area $= \left(\frac{d_1d_2}{2} + \frac{d_1d_3}{2} + \frac{d_2d_3}{2}\right) \sin 120°$

$$= \frac{\sqrt{3}}{4} \,.\, (d_1d_2 + d_1d_3 + d_2d_3).$$

35. Let $OA=a$, $OB=b$, $OC=c$; $\angle OBA=\theta$, and let x be the side of the square $ABCD$.

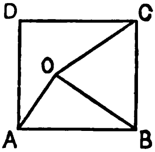


FIG. 79.

$$\text{Then } \angle OBC=90^\circ-\theta,$$

$$\text{and } a^2=x^2+b^2-2bx\cos\theta;$$

$$c^2=x^2+b^2-2bx\sin\theta;$$

$$\therefore\ 2bx\cos\theta=x^2+b^2-a^2,$$

$$2bx\sin\theta=x^2+b^2-c^2.$$

Squaring and adding these equations,

$$4b^2x^2=x^4+2(b^2-a^2)x^2+(b^2-a^2)^2+x^4+2(b^2-c^2)x^2+(b^2-c^2)^2;$$

$$\therefore\ 2x^4-2(a^2+c^2)x^2+(a^2+c^2)^2=2(a^2b^2+a^2c^2+b^2c^2-b^4),$$

$$\text{and } x=\sqrt{\frac{1}{2}\left\{a^2+c^2\pm\sqrt{4(a^2b^2+a^2c^2+b^2c^2-b^4)-(a^2+c^2)^2}\right.}.$$

(Gaskin's *Solutions of Trigonometrical Examples.*)

36. Let ABC be any triangle described about a circle.

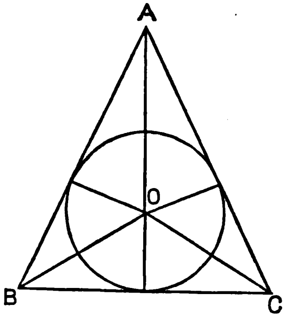


FIG. 80.

Then area of ABC=area of AOB+area of BOC+area of AOC.

$$\therefore\ \text{area of } ABC=\frac{1}{2}\,.\,rc+\frac{1}{2}ra+\frac{1}{2}rb.$$

$$=\frac{r}{2}(a+b+c);$$

$\therefore$ since r is constant,

area of $ABC \propto (a+b+c)$.

37.
$$\alpha = AD = c\,.\,\sin B = b\,.\,\sin C,$$
$$\beta = BE = c\,.\,\sin A,$$
$$\gamma = CF = b\,.\,\sin A\,;$$
$$\therefore\ \frac{a^2}{\beta\gamma} = \frac{bc\,.\,\sin B\,.\,\sin C}{bc\,.\,\sin A\,.\,\sin A} = \frac{bc}{a^2}.$$

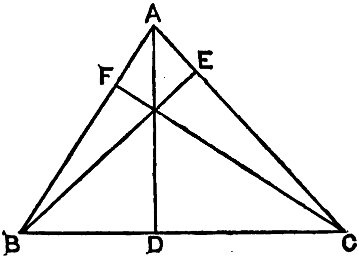


FIG. 81.

$$\text{Similarly } \frac{\beta^2}{\alpha\gamma} = \frac{ac}{b^2}\,;\ \text{and } \frac{\gamma^2}{\alpha\beta} = \frac{ab}{c^2}\,;$$
$$\therefore\ \frac{\alpha^2}{\beta\gamma} + \frac{\beta^2}{\alpha\gamma} + \frac{\gamma^2}{\alpha\beta} = \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2}.$$

38. Let A be the observer on the sea-shore, O the earth's centre, BC the mountain whose height = 1284·8 yards = ·73 miles.

Then since C is just visible from A,

AC is a tangent at A.

Join OA and produce it to D, making $AD = 3$ miles; then $\angle DCA$ = angle of depression of C from $D = 2°.15'$.

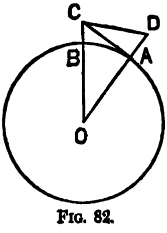


FIG. 82.

$$\text{Then } AC = 3\,.\,\cot 2°.15'$$
$$\log AC = \log 3 + L\cot 2°.15' - 10$$
$$= \cdot 4771213 + 11\cdot 4057168 - 10$$
$$= 1\cdot 8828381\,;$$
$$\therefore\ AC = 76\cdot 3551.$$

Let OA, the earth's radius, $=r$;

$$\therefore\ AC^2=BC(2r+BC)=\cdot73(2r+\cdot73),$$

and $\log(2r+\cdot73)=2\log AC-\log\cdot73=3\cdot9023533$;

$$\therefore\ 2r+\cdot73=7986\cdot4;$$

$$\therefore\ r=3992\cdot835 \text{ miles.}$$

(Gaskin's *Solutions of Trigonometrical Examples.*)

39. Let ABC be the triangle, $CO=b$, $BO=a$,

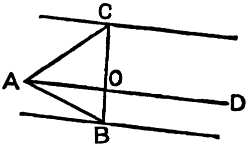


FIG. 83.

$\angle BAD=\theta$, and $\therefore\ \angle CAD=60^\circ-\theta$.

Let $AB=x$.

Then $x.\sin\theta=a$,

$x\sin(60^\circ-\theta)=b$;

$$\therefore\ \frac{\sin(60^\circ-\theta)}{\sin\theta}=\frac{b}{a}.$$

$$\therefore\ \frac{\sqrt{3}}{2}\cdot\cot\theta-\frac{1}{2}=\frac{b}{a}.$$

$$\therefore\ a.\cot\theta=\frac{3b+a}{\sqrt{3}}.$$

$$\therefore\ x=a.\operatorname{cosec}\theta=\sqrt{a^2+\frac{4b^2+4ab+a^2}{3}}=2\sqrt{\frac{a^2+ab+b^2}{3}}$$ (Gaskin).

40.

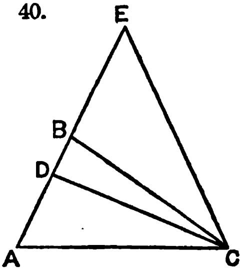


FIG. 84.

$$\frac{AD}{DB}=\frac{AC}{CB}=2;\ \therefore\ AD=2DB,$$

$$AB=AD+DB=3DB,$$

$$\frac{AE}{EB}=\frac{AC}{CB}=2;\ \therefore\ AE=2BE;$$

$$\therefore\ BE=AB=3DB;$$

$$\therefore\ DE=BE+DB=4DB.$$

Then, by EUCLID, VI. i.

$$\triangle CBD : \triangle ACD : \triangle ABC : \triangle CDE$$

$$=DB : AD : AB : DE$$

$$=1\ \ :2\ \ :3\ \ :4.$$ (Gaskin).

41. $$R.\sin A=\frac{a}{2}\text{, by Art. 221;}$$

$$\therefore\ Rr.(\sin A+\sin B+\sin C)$$

$$=r.\left(\frac{a+b+c}{2}\right)$$

$$=r.s$$

$$=\text{area of the triangle.}$$

42. The circles have the same radius because $R=\frac{b}{2\sin B}$.

In the example given, $\sin 50^\circ.15'=\cdot 7688418$;

$$\therefore\ R=\frac{564}{1\cdot 5376836}=366\cdot 785.$$

43. Call the angles $x, \frac{x+y}{2}, \frac{x+2y}{2}, \frac{x+3y}{3}$.

$$\left.\begin{aligned}&\text{Then } x+\frac{x+y}{2}+\frac{x+2y}{2}+\frac{x+3y}{3}=2\pi\\&\text{and } x+\frac{x+2y}{2}=\pi\end{aligned}\right\};$$

$$\therefore\ \left.\begin{aligned}14x+15y&=12\pi\\3x+2y&=2\pi\end{aligned}\right\}.$$

$$\text{Hence } x=\frac{6\pi}{17}\text{ and } y=\frac{8\pi}{17};$$

$$\therefore\ \text{the angles are } \frac{6\pi}{17}, \frac{7\pi}{17}, \frac{11\pi}{17}, \frac{10\pi}{17}$$

44. $$\frac{(1+\cot PCA)^2}{(1+\cot PCB)^2}=\frac{\left(1-\frac{CM}{PM}\right)}{\left(1+\frac{CM}{PM}\right)^2}$$

$$=\frac{(PM-CM)^2}{(PM+CM)^2}$$

$$=\frac{CP^2-2CN.PN}{CP^2+2CN.PN}$$

$$=\frac{CN.CD-2CN.PN}{CN.CD+2CN.PN}$$

$$=\frac{CO-PN}{CO+PN}=\frac{CB-CM}{AC+CM}=\frac{BM}{AM}=\frac{\cot PBA}{\cot PAB}.$$

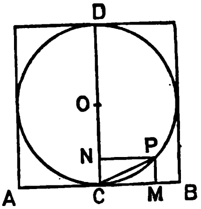


FIG. 85.

45. $$r+r_a+r_b-r_c=\frac{S}{s}+\frac{S}{s-a}+\frac{S}{s-b}-\frac{S}{s-c}$$
$$=\frac{S(2s-a)}{s.(s-a)}+\frac{S.(s-c-s+b)}{(s-b)(s-c)}$$
$$=\frac{S.(b+c)}{s.(s-a)}+\frac{S.(b-c)}{(s-b)(s-c)}$$
$$=S.\left\{\frac{b.\{(s-b)(s-c)+s.(s-a)\}+c\{(s-b)(s-c)-s.(s-a)\}}{S^2}\right\}$$
$$=\frac{b\{2s^2-s.(a+b+c)+bc\}+c\{-s(b+c-a)+bc\}}{S}$$
$$=\frac{b^2c-\frac{c}{2}(b+c+a)(b+c-a)+bc^2}{S}$$
$$=\frac{c}{2S}\cdot\left\{2b^2-(b+c)^2+a^2+2bc\right\}$$
$$=\frac{c}{2S}(b^2-c^2+a^2)=\frac{c}{2S}2ab\cos C=\frac{abc.\cos C}{S}=4R.\cos C.$$

46. Let C be the right angle; then, by Art. 223,

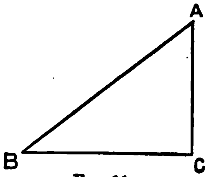


FIG. 86.

$$S=\frac{c+b-a}{2}\cdot r,\text{ and}$$
$$S=\frac{c+a-b}{2}\cdot r';$$
$$\therefore S^2=\frac{c^2-(a-b)^2}{4}\cdot rr'$$
$$=\frac{c^2-a^2+2ab-b^2}{4}\cdot rr'=\frac{ab}{2}rr'=S.rr';$$
$$\therefore rr'=S.$$

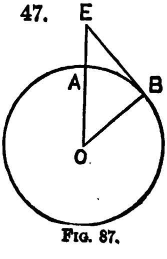
47. 

FIG. 87.

Using the notation of Art. 228,

$OB=4000$ miles,

$OE=OB.\sec.1°.58'.10''$

$=4000\times1{\cdot}005910$

$=4002{\cdot}364.$

$\therefore AE=2{\cdot}36\ldots$ miles.

48. Using the notation of Art. 228,

$$\sec EOB = \frac{4001{\cdot}25}{4000} = 1{\cdot}0003125,$$

and, by the Tables, $\sec 1^\circ.\, 26' = 1{\cdot}0003130$.

Hence dip of horizon $= 1^\circ.\, 26'$ nearly.

49. Let A be the man's eye; B the lamp; C the centre of the earth.

Then $AD + DB = 52800$ feet.

And, if the radius of the earth be R feet,

$$AD^2 = 6(2R + 6),$$
$$BD^2 = 32(2R + 32).$$

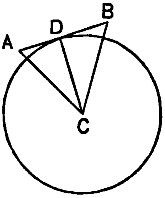


FIG. 88.

Hence, approximately, $\sqrt{12R} + \sqrt{64R} = 52800$,

or, $\sqrt{R}\,.\,(4 + \sqrt{3}) = 26400$, or, $\sqrt{R} \times 13 = 26400(4 - \sqrt{3})$;

$$\therefore\ R = \frac{26400 \times 26400 \times (19 - 8\sqrt{3})}{13 \times 13 \times 1760 \times 3}\text{miles} = 4017{\cdot}79\ldots\text{ miles.}$$

50. In 72 minutes the ship travels 12 miles.

Then using the notation of Art. 228,

$$BE^2 = CE\,.\,EA,$$

$$144 = \left(CA + \frac{90}{5280}\right)\cdot\frac{90}{5280}$$

$$= CA\,.\,\frac{90}{5280}\text{ nearly};$$

$$\therefore\ CA = \frac{144 \times 528}{9} = 16 \times 528 = 8448.$$

$\therefore$ radius $= 4224$ miles.

51. Using the notation of Art. 228,

$$\cos EOB=\frac{OB}{OE}=\frac{3956}{3959}.$$

$$\therefore\ L\cos EOB=10+\log 3956-\log 3959$$

$$=10+3{\cdot}5972563-3{\cdot}5975855$$

$$=9{\cdot}9996708.$$

Whence, by the tables,

$$EOB=2°.\ 13'.\ 50''.$$

52. Let r be the radius of a section of the earth, made by a plane through its centre perpendicular to the line joining its centre with the sun's centre. Then if θ be the circular measure of the angle subtended by r at the sun's centre, and d be the distance between the two centres,

$$\frac{r}{d}=\tan\theta=\theta \text{ nearly, since } \theta \text{ is very small.}$$

$$\therefore\ \frac{r}{d}=\frac{8{\cdot}868}{57{\cdot}29577\times 60\times 60}.$$

$$\therefore\ d=\frac{57{\cdot}29577\times 60\times 60\times 4000}{8{\cdot}868}$$

$$=\frac{206264772}{2{\cdot}217}=93037786{\cdot}1\ .\ .\ .\ \text{miles.}$$

53. Using the same notation as in Ex. 52,

$$\tan\theta=\frac{4000}{241118};$$

$$\therefore\ L\tan\theta=10+\log 4000-\log 241118$$

$$=10+3{\cdot}6020600-5{\cdot}3822296$$

$$=8{\cdot}2198304.$$

Hence, by the tables,

$$\theta=57'.\ 1''{\cdot}5=\text{nearly.}$$

54. Let A, B be the two points; then AB is a tangent at its middle point D to the earth's surface.

$$AD = DE \text{ nearly} = 4 \text{ miles},$$

$$AE = 10 \text{ feet} = \frac{10}{5280} \text{miles}.$$

Let C be the earth's centre, and $CD = r$.

$$\text{Then } AE(2r + AE) = AD^2.$$

$$\therefore, \text{ approximately, } AE \,.\, 2r = AD^2;$$

$$\therefore\ r = \frac{16 \times 5280}{10 \times 2} = 4224 \text{ miles}.$$

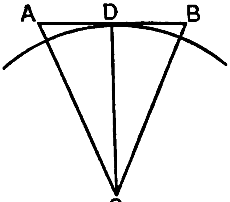


FIG. 89.

55. The limit of deviation is the angle subtended by the radius of the target at a point 600 feet distant, and if this angle be denoted by θ

$$\tan\theta = \frac{2}{600};$$

$$\therefore\ \theta = \tan^{-1}\cdot 00\dot{3}.$$

56. Regard the moon M as the base of a cone of which E, the eye of the observer, is the vertex. Then S, the shilling, will intercept all the rays of light from M to E, when it is so near to S that lines from E to the circumference of S do not, when produced, fall within the circumference of M.

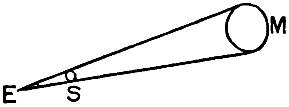


FIG. 90.

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[*In preparation.*

Battle of the Herrings.

these too far apart, and the intercourse of the defenders with an army of relief under the Count of Clermont at Blois was not broken off. Early in the following year, this army hoped to raise the siege by falling on a large body of provisions coming to the besiegers from Paris under Sir John Fastolf. The attack was made at Rouvray, but Fastolf had made careful preparations. The waggons were arranged in a square, and, with the stakes of the archers, formed a fortification on which the disorderly attack of the French made but little impression. Broken in the assault, they fell an easy prey to the English, as they advanced beyond their lines. The skirmish is known by the name of the Battle of the Herrings. This victory, which deprived the besieged of hope of external succour, seemed to render the capture of the city certain.

Danger of Orleans.

Already at the French King's court at Chinon there was talk of a hasty withdrawal to Dauphiné, Spain, or even Scotland; when suddenly there arose one of those strange effects of enthusiasm which sometimes set all calculation at defiance.

Joan of Arc.

In Domrémi, a village belonging to the duchy of Bar, the inhabitants of which, though in the midst of Lorraine, a province under Burgundian influence, were of patriotic views, lived a village maiden called Joan of Arc. The period was one of great mental excitement; as in other times of wide prevailing misery, prophecies and mystical preachings were current. Joan of Arc's mind was particularly susceptible to such influences, and from the time she was thirteen years old, she had fancied that she heard voices, and had even seen forms, sometimes of the Archangel Michael, sometimes of St. Catherine and St. Margaret, who called her to the assistance of the Dauphin. She persuaded herself that she was destined to fulfil an old prophecy which said that the kingdom, destroyed by a woman—meaning, as she thought, Queen Isabella,—should be saved by a maiden of Lorraine. The burning of Domrémi in the summer of 1428 by a troop of Burgundians at length gave a practical form to her imaginations, and early in the following year she succeeded in persuading Robert of Baudricourt to send her, armed and accompanied by a herald, to Chinon. She there, as it is said by the wonderful knowledge she displayed, convinced the court of the truth of her mission. At all events, it was thought wise to take advantage of the infectious enthusiasm she displayed, and in April she was intrusted with an army of 6000 or 7000 men, which was to march up the river from Blois to the relief of Orleans. When she appeared upon the scene of war, she supplied exactly that element of success of all of them open by two slits turned towards the centre of the flower. Their stalks have expanded and joined together, so as to form a thin sheath round the central column (fig. 12). The dust-spikes are so variable in length in this flower, that it may not be possible to see that one short one comes between two long ones, though this ought to be the case.

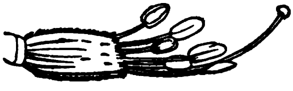


Fig. 12.
Dust-spikes of gorse (*enlarged*).

The *seed-organ* is in the form of a longish rounded pod, with a curved neck, stretching out beyond the dust-spikes. The top of it is sticky, and if you look at a bush of gorse, you will see it projecting beyond the keel in most of the fully-blown flowers, because the neck has become more curved than in fig. 12. Cut open the pod; it contains only one cavity (not, as that of the wall-flower, two separated by a thin partition), and the grains are suspended by short cords from the top (fig. 13). These grains may be plainly seen in the seed-organ of even a young flower. It is evident that they are the most important part of the plant, as upon them depends its diffusion and multiplication. We have already seen how carefully their well-being is considered in the matter of their perfection, how even insects are pressed into their service for this purpose! Now let us glance again at our flower, and see how wonderfully contrivance is heaped upon contrivance for their protection!

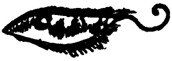


Fig. 13.
Split seed-pod of gorse.

First (see fig. 10, p. 14), we have the outer covering, so covered with hairs, that it is as good for keeping out rain as a waterproof cloak; in the buttercup, when you pressed the bud, it separated into five leaves; here there are five leaves, just the same, but they are so tightly joined that you may press till the whole bud is bent without making them separate at all, and when the bud is older, they only separate into two, and continue to enfold the flower to a certain extent till it fades. When the flower pushes back its waterproof cloak, it has the additional shelter of the big struction, and at last, after nearly twenty years of alternate hopes and fears, of tedious negotiations, official evasions, and sterile Parliamentary debates, it was effectually extinguished by the adverse report of a Parliamentary Committee, followed by the erection of the present Millbank Penitentiary at a vastly greater expense and on a totally different system.

Transportation.—In the meantime the common gaols were relieved in a makeshift fashion by working gangs of prisoners in hulks at the seaports; but the resource mainly relied on for getting rid of more dangerous criminals was the old one of transportation, Botany Bay having succeeded to America. As at first employed, there was no mistake as to the reality of the punishment; the misfortune was that the worst elements in the real were not so made known as to form any part of the apparent punishment. If the judge, in sentencing the convict, had thought fit to explain, for the warning of would-be offenders, exactly what was going to be done with their associate, the sentence would have been something of this sort: "You shall first be kept, for days or months as it may happen, in a common gaol, or in the hulks, in company with other criminals better or worse than yourself, with nothing to do, and every facility for mutual instruction in wickedness. You shall then be taken on board ship with similar associates of both sexes, crammed down between decks, under such circumstances that about one in ten of you will probably die in the course of the six months' voyage. If you survive the voyage you will either be employed as a slave in some public works, or let out as a slave to some of the few free settlers whom we have induced to go out there. In either case you will be under very little regular inspection, and will have every opportunity of indulging those natural wealth into the treasury. Churches remained open day and night, and frequent addresses kept up the enthusiasm to a high pitch. It was (for the moment) a genuine "revival" or reawakening of the whole Roman world. The occasion, too, appeared favourable. Italy was quiet, and the Exarchate at peace with its neighbours. Clotaire the Frank was no enemy to Heraclius, and in common with his clergy (being orthodox and not Arian) might be expected to sympathise in so holy a cause.

Treachery of the Avars—A.D. 616.—In one quarter only was there room for fear. The Avars were on the Danube, and the turbulence of the Avars was only equalled by their perfidy. Already, in A.D. 610, they had fallen suddenly on North Italy, and pillaged and harassed those same Lombards whom they had before helped to destroy the Gepidæ. Previous to an absence, therefore, of years from his capital, it was essential for the Emperor to sound their intentions, and, if possible, to secure their neutrality. His ambassadors were welcomed with apparent cordiality, and an interview was arranged between the Chagan and Heraclius. The place was to be Heraclea. At the appointed time the Emperor set out from Selymbria to meet the Khan, decked with Imperial crown and mantle to honour the occasion. The escort was a handful of soldiers; but there was an immense cortége of high officials and of the fashionable world of Constantinople, and the whole country side was there to see. Presently some terrified peasants were seen making their way hurriedly towards Heraclius. They urged him to flee for his life; for armed Avars had been seen in small bodies, and might even now be between him and the capital. Heraclius knew too much to hesitate. He threw off his robes and fled, and but just in time. The Chagan had laid a deep plot. A large mass of men had been told off in small detachments

I say the pulpit (in the sober use
Of its legitimate peculiar pow'rs)
Must stand acknowledg'd, while the world shall stand,
The most important and effectual guard,
Support and ornament of virtue's cause.
There stands the messenger of truth: there stands
The legate of the skies; his theme divine,
His office sacred, his credentials clear.
By him, the violated law speaks out
Its thunders, and by him, in strains as sweet
As angels use, the Gospel whispers peace.
He stablishes the strong, restores the weak,
Reclaims the wand'rer, binds the broken heart,
And, arm'd himself in panoply complete
Of heav'nly temper, furnishes with arms
Bright as his own, and trains, by ev'ry rule
Of holy discipline, to glorious war,
The sacramental host of God's elect.
Are all such teachers? would to heav'n all were!
But hark—the Doctor's voice—fast wedged between
Two empirics he stands, and with swoln cheeks
Inspires the news, his trumpet. Keener far
Than all invective is his bold harangue,
While through that public organ of report
He hails the clergy; and, defying shame,
Announces to the world his own and theirs.
He teaches those to read, whom schools dismiss'd,
And colleges, untaught; sells accent, tone,
And emphasis in score, and gives to pray'r
Th' *adagio* and *andante* it demands.
He grinds divinity of other days
Down into modern use; transforms old print
To zigzag manuscript, and cheats the eyes
Of gall'ry critics by a thousand arts.—
Are there who purchase of the Doctor's ware?
Oh name it not in Gath!—it cannot be,
That grave and learned Clerks should need such aid.
He doubtless is in sport, and does but droll,
Assuming thus a rank unknown before,
Grand caterer and dry-nurse of the church.

I venerate the man whose heart is warm,
Whose hands are pure, whose doctrine and whose life.

[COWPER'S TASK—*See Page* 6.]

[*Specimen Page, No.* 6.]

gether as with a close seal. . . . . The flakes of his flesh are joined together: they are firm in themselves; they cannot be moved."

Hobbes, in his famous book to which he gave the title *Leviathan*, symbolised thereby the force of civil society, which he made the foundation of all right.

315–325 Cowper's limitation of the province of satire—that it is fitted to laugh at foibles, not to subdue vices—is on the whole well-founded. But we cannot forget Juvenal's famous "facit indignatio versum," or Pope's no less famous—

> "Yes, I am proud: I must be proud to see
> Men not afraid of God, afraid of me:
> Safe from the bar, the pulpit, and the throne,
> Yet touched and shamed by ridicule alone."

326–372 *The pulpit, not satire, is the proper corrector of sin. A description of the true preacher and his office, followed by one of the false preacher, "the reverend advertiser of engraved sermons."*

330 *Strutting and vapouring.* Cf. *Macbeth*, v. 5.

> "Life 's but a walking shadow, a poor player,
> That struts and frets his hour upon the stage,
> And then is heard no more; it is a tale
> Told by an idiot, full of sound and fury,
> Signifying nothing."

> "And what in real value 's wanting,
> Supply with vapouring and ranting."—HUDIBRAS.

331 *Proselyte.* προσήλυτος, a new comer, a convert to Judaism.

338 *His theme divine.* Nominative absolute.

343 *Stablishes.* Notice the complete revolution the word has made—stabilire, établir, establish, stablish; cf. state, &c.

346 *Of heavenly temper.* Cf. *Par. Lost*, i. 284, "his ponderous shield etherial temper." See note on *Winter Morning Walk*, l. 664.

349 *Sacramental.* Used in the Latin sense. Sacramentum was the oath of allegiance of a Roman soldier. The word in its Christian sense was first applied to baptism—the vow to serve faithfully under the banner of the cross. See *Browne on the Thirty-nine Articles*, p. 576.

350 *Would to heaven.* A confusion between "would God" and "I pray to heaven."

351 A picture from the life of a certain Dr Trusler, who seems to have combined the trades of preacher, teacher of elocution, writer of sermons, and literary hack.

352 *Empirics.* ἐμπειρικός, one who trusts solely to experience or practice instead of rule, hence a quack. The accent is the same as in Milton (an exception to the rule. See note on *Sofa*, l. 52).

[COWPER'S TASK—*See Page* 6.]

thus: if the articles had cost £1 each, the total cost would have been £2478;

∴ as they cost $\frac{1}{6}$ of £1 each. the cost will be £$\frac{2478}{6}$, or £413.

The process may be written thus:

3*s*. 4*d*. is $\frac{1}{6}$ of £1	£2478 = cost of the articles at £1 each.
	£413 = cost .... at 3*s*. 4*d*. ...

Ex. (2). Find the cost of 2897 articles at £2. 12*s*. 9*d*. each.

£2 is 2 × £1	2897 . 0 . 0	= cost at £1 each.
10*s*. is $\frac{1}{2}$ of £1	5794 . 0 . 0	= ...... £2 ....
2*s*. is $\frac{1}{5}$ of 10*s*.	1448 . 10 . 0	= ...... 10*s*.....
8*d*. is $\frac{1}{3}$ of 2*s*.	289 . 14 . 0	= ...... 2*s*. ....
1*d*. is $\frac{1}{8}$ of 8*d*.	96 . 11 . 4	= ...... 8*d*. ....
	12 . 1 . 5	= ...... 1*d*. ....
	£7640 . 16 . 9	= ...... £2. 12*s*. 9*d*. each.

NOTE.—A shorter method would be to take the parts thus:

10*s*. = $\frac{1}{2}$ of £1; 2*s*. 6*d*. = $\frac{1}{4}$ of 10*s*.; 3*d*. = $\frac{1}{10}$ of 2*s*. 6*d*.

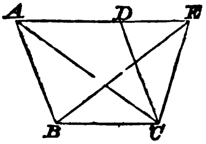
Ex. (3). Find the cost of 425 articles at £2. 18*s*. 4*d*. each.

Since £2. 18*s*. 4*d*. is the difference between £3 and 1*s*. 8*d*. (which is $\frac{1}{12}$ of £1), the shortest course is to find the cost at £3 each, and to *subtract from it* the cost at 1*s*. 8*d*. each, thus:

	£ *s*. *d*.	
£3 is 3 × £1	425 . 0 . 0	= cost at £1 each.
1*s*. 8*d*. is $\frac{1}{12}$ of £1	1275 . 0 . 0	= ...... £3 ....
	35 . 8 . 4	= ...... 1*s*. 8*d*. each.
	£1239 . 11 . 8	= ...... £2. 18*s*. 4*d*. each.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.



Let the ▱ $ABCD$ and the $\triangle EBC$ be on the same base BC and between the same ||s AE, BC.

Then must ▱ $ABCD$ *be double of* $\triangle$ EBC.

Join AC.

Then $\triangle ABC = \triangle EBC$, ∵ they are on the same base and between the same ||s ; I. 37.

and ▱ $ABCD$ is double of $\triangle ABC$, ∵ AC is a diagonal of $ABCD$; I. 34.

∴ ▱ $ABCD$ is double of $\triangle$ EBC.

Q. E. D.

Ex. 1. If from a point, without a parallelogram, there be drawn two straight lines to the extremities of the two opposite sides, between which, when produced, the point does not lie, the difference of the triangles thus formed is equal to half the parallelogram.

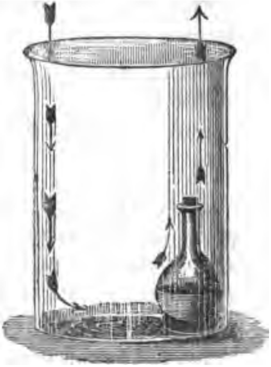
Ex. 2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

Sometimes carbonic anhydride is produced in wells, and, being so much heavier than air, it remains at the bottom. If a man goes down into such a well, he will have no difficulty at first, because the air is good; but when he is near the bottom, where the gas has accumulated, he will gasp for breath and fall; and if anyone, not understanding the cause of his trouble, goes down to assist him, he too will fall senseless, and both will quickly die. The way to ascertain whether carbonic anhydride has accumulated at the bottom of a well is to let a light down into it. If it goes out, or even burns very dimly, there is enough of the gas to make the descent perilous. A man going down a well should always take a candle with him, which he should hold a considerable distance below his mouth.' If the light burns dimly, he should at once stop, before his mouth gets any lower and he takes some of the gas into his lungs.

When this gas is in a well or pit, of course it must be expelled before a man can descend. There are several expedients for doing this. One is to let a bucket down frequently, turning it upside down, away from the mouth of the well, every time it is brought up, a plan which will remind you of the experiment represented in Fig. 24.

But a better way is to let down a bundle of burning straw or shavings, so as to heat the gas. Now heated bodies expand, gases very much more than solids or liquids, and, in expanding, the weight of a certain volume, say of a gallon, becomes lessened. So that if we can heat the carbonic anhydride enough to make a gallon of it weigh less than a gallon of air, it will rise out of the well just as hydrogen gas would do. Fig. 25 shows how you may perform this experiment upon a small scale.

Fig. 25.



DISASTROUS RETREAT OF THE ENGLISH FROM CABUL.

IT took two days of disorder, suffering, and death to carry the army, now an army no more, to the jaws of the fatal pass. Akbar Khan, who appeared like the Greeks' dread marshal from the spirit-land at intervals upon the route, here demanded four fresh hostages. The demand was acquiesced in. Madly along the narrow defile crowded the undistinguishable host, whose diminished numbers were still too numerous for speed: on every side rang the war-cry of the barbarians: on every side plundered and butchered the mountaineers: on every side, palsied with fatigue, terror, and cold, the soldiers dropped down to rise no more. The next day, in spite of all remonstrance, the general halted his army, expecting in vain provisions from Akbar Khan. That day the ladies, the children, and the married officers were given up. The march was resumed. By the following night not more than one-fourth of the original number survived. Even the haste which might once have saved now added nothing to the chances of life. In the middle of the pass a barrier was prepared. There twelve officers died sword in hand. A handful of the bravest or the strongest only reached the further side alive: as men hurry for life, they hurried on their way, but were surrounded and cut to pieces, all save a few that had yet escaped. Six officers better mounted or more fortunate than the rest, reached a spot within sixteen miles of the goal; but into the town itself rode painfully on a jaded steed, with the stump of a broken sword in his hand, but one.

Livy, xxi. c. 25, § 7-10. xxxv. c. 30. xxiii. c. 24.
Cæsar, *Bell. Gall.* v. c. 35-37.

DEFEAT OF CHARLES THE BOLD AND MASSACRE OF HIS TROOPS AT MORAT.

IN such a predicament braver soldiers might well have ceased to struggle. The poor wretches, Italians and Savoyards, six thousand or more in number, threw away their arms and made

[*Specimen Page, No.* 11.]

II.

ARIADNE'S LAMENT.

Madam, 'twas Ariadne passioning
For Theseus' perjury and unjust flight.
TWO GENTLEMEN OF VERONA, IV. 4, 172.

ARGUMENT.

ARIADNE *tells the story of her first waking, to find herself abandoned by Theseus and left on an unknown island, exposed to a host of dangers.*—(HEROIDES, X.)

The story is beautifully told by Catullus, in the "Epithalamium Pelei et Thetidos:" it also forms one of the episodes in Chaucer's "Legende of Goode Women."

I woke before it was day to find myself alone, no trace of my companions to be seen. In vain I felt and called for Theseus; the echoes alone gave me answer.

QUAE legis, ex illo, Theseu, tibi litore mitto,
  Unde tuam sine me vela tulere ratem:
In quo me somnusque meus male prodidit et tu,
  Per facinus somnis insidiate meis. 107
Tempus erat, vitrea quo primum terra pruina 112
  Spargitur et tectae fronde queruntur aves:
Incertum vigilans, a somno languida, movi 97
  Thesea prensuras semisupina manus:
Nullus erat, referoque manus, iterumque retempto,
  Perque torum moveo brachia: nullus erat.
Excussere metus somnum: conterrita surgo,
  Membraque sunt viduo praecipitata toro. 123
Protinus adductis sonuerunt pectora palmis, 111
  Utque erat e somno turbida, rapta coma est.
Luna fuit: specto, siquid nisi litora cernam;
  Quod videant, oculi nil nisi litus habent. 150
Nunc huc, nunc illuc, et utroque sine ordine curro;
  Alta puellares tardat arena pedes.
Interea toto clamanti litore "Theseu!" 121
  Reddebant nomen concava saxa tuum,
Et quoties ego te, toties locus ipse vocabat:
  Ipse locus miserae ferre volebat opem. 106 3

[TAYLOR'S OVID—*See Page* 16.]

174. **Punica poma,** pomegranates.
178. **Taenarum,** at the southern extremity of Peloponnesus, was one of the numerous descents to Tartarus. Cf. Virgil, Georg. IV. 467:

> Taenarias etiam fauces, alta ostia Ditis.

179. **Factura fuit.** This periphrasis for *fecisset* is to be noted; it is the one from which the oblique forms are all constructed, *e.g.*, *facturam fuisse*, or *factura fuisset*.
183. **Cessatis,** one of a goodly number of intransitive verbs of the first conjugation which have a passive participle. Cf. **erratas,** above, 139, **clamata,** 35. So Horace, regnata Phalanto rura (Odes, II. 6, 12); triumphatae gentes (Virgil).

II.—IV.

ARIADNE.

THIS and the two following extracts, though taken from different works, form a definite sequence. Ariadne, daughter of Minos, king of Crete, has helped Theseus to conquer the Minotaur, by giving him a clew to the maze in which the monster was hid, and, being in love with him, has fled in his company. They put in for the night to the island of Dia, and Theseus on the next morning treacherously sails away, leaving the poor girl alone. The first extract is part of an epistle which she is supposed to write on the day when she discovers his perfidy.

The name Dia, which belonged properly to a small island off the north coast of Crete, was also a poetical name for Naxos, one of the largest of the Cyclades. It may have been this fact which led to the further legend which is recounted in the next extract, how Ariadne, lorn of Theseus, becomes the bride of Bacchus; for Naxos was the home of the Bacchic worship. As the completion of the legend she is raised to share in Bacchus' divine honours, and as the Cretan Crown becomes one of the signs of the heavens.

II.

ARIADNE'S LAMENT.

1. **Illo,** sc. *Diae.*
4. **Per facinus,** criminally.
5. Describing apparently the early dawn, or the hour that precedes it, when the night is at its coldest, and the birds, half-awake, begin to stir in their nests. **Pruina** hints that it is autumn.
7. A beautifully descriptive line—But half-awake, with all the languor of sleep still on me.

   A somno = after, as the *result* of.
8. **Semisupina,** on my side, lit., half on my back, describes the motion of a person thus groping about on waking. Cf. Chaucer:

> Ryght in the dawenynge awaketh shee,
> And gropeth in the bed, and fonde ryghte noghte.

haec mea magna fides? at non, Euandre, pudendis
volneribus pulsum aspicies, nec sospite dirum
optabis nato funus pater. ei mihi, quantum
praesidium Ausonia, et quantum tu perdis, Iule!
Haec ubi deflevit, tolli miserabile corpus
imperat, et toto lectos ex agmine mittit
mille viros, qui supremum comitentur honorem,
intersintque patris lacrimis, solacia luctus
exigua ingentis, misero set debita patri.
haut segnes alii crates et molle feretrum
arbuteis texunt virgis et vimine querno,
extructosque toros obtentu frondis inumbrant.
hic iuvenem agresti sublimem stramine ponunt;
qualem virgineo demessum pollice florem
seu mollis violae, seu languentis hyacinthi,
cui neque fulgor adhuc, nec dum sua forma recessit;
non iam mater alit tellus, viresque ministrat.
tunc geminas vestes auroque ostroque rigentis
extulit Aeneas, quas illi laeta laborum
ipsa suis quondam manibus Sidonia Dido
fecerat, et tenui telas discreverat auro.
harum unam iuveni supremum maestus honorem
induit, arsurasque comas obnubit amictu;
multaque praeterea Laurentis praemia pugnae
aggerat, et longo praedam iubet ordine duci.
addit equos et tela, quibus spoliaverat hostem.
vinxerat et post terga manus, quos mitteret umbris
inferias, caeso sparsuros sanguine flammam;
indutosque iubet truncos hostilibus armis
ipsos ferre duces, inimicaque nomina figi.
ducitur infelix aevo confectus Acoetes,
pectora nunc foedans pugnis, nunc unguibus ora;
sternitur et toto proiectus corpore terrae.

[*Specimen Page, No.* 14.]

Comp. *Geor.* ii. 80, *Nec longum tempus et . . . exiit . . . arbos*, C. But as these are the only two instances of the construction adduced it is perhaps safer to take *et* = even.

51 **nil iam,** etc.] The father is making vows to heaven in his son's behalf, but the son is gone where vows are neither made nor paid.

55 **haec mea magna fides**] 'Is this the end of all my promises?' *Magna* may be taken as 'solemn,' or 'boastful.'

pudendis volneribus] All his wounds are on his breast.

56 **dirum optabis funus** = *morti devovebis.* Compare the meaning of *dirae*, xii. 845.

59-99] A description of the funeral rites. Aeneas bids his last farewell.

59 **Haec ubi deflevit**] 'His moan thus made.' *De* in composition has two opposite meanings: (1) cessation from or removal of the fundamental ideas, as in *decresco*, *dedoceo*, etc.; (2) (as here) in intensifying, as *debello*, *demiror*, *desaevio*.

61 **honorem**] *Honos* is used by V. for (1) a sacrifice, iii. 118; (2) a hymn, *Geor.* ii. 393; (3) beauty, *Aen.* x. 24; (4) the 'leafy honours' of trees, *Geor.* ii. 404; (5) funeral rites, vi. 333, and here. See below, *l.* 76.

63 **solatia**] In apposition to the whole sentence; whether it is nom. or acc. depends on how we resolve the principal sentence; here, though *solatia* applies to the whole sentence, its construction probably depends on the last clause, which we may paraphrase, *ut praesentes* (τὸ μετεῖναι) *sint solatia ;* therefore it is nom.

64 **crates et molle feretrum**] The bier of pliant osier: cf. *l.* 22.

66] Cf. Statius, *Theb.* vi. 55, *torus* et puerile feretrum.

obtentu frondis] 'A leafy canopy.' C. understands 'a layer of leaves.

67 **agresti stramine**] 'The rude litter.'

68] Cf. ix. 435; *Il.* viii. 306,

> μήκων δ' ὡς ἑτέρωσε κάρη βάλεν, ἥτ ἐνὶ κήπῳ
> καρπῷ βριθομένη νοτίῃσί τε εἰαρινῇσιν·
> ὣς ἑτέρωσ' ἤμυσε κάρη πήληκι βαρυνθέν.

> 'Even as a flower,
> Poppy or hyacinth, on its broken stem
> Languidly raises its encumbered head.'—MILMAN.

69 **languentis hyacinthi**] The rhythm is Greek. The 'drooping hyacinth' is probably the Lilium Martagon or Turk's-cap lily, 'the sanguine flower inscribed with woe.'

70] 'That hath not yet lost its gloss nor all its native loveliness.' *Recessit* must apply to both clauses. 'If we suppose the two parts of the line to contain a contrast, the following line will lose much of its force,' C. Compare the well-known lines from the *Giaour*, 'He who hath bent him o'er the dead,' etc.

71] Contrast the force of *neque adhuc*, *nec dum*, and *non iam ;* 'the brightness not all gone,' 'the lines where beauty lingers,' and 'the support and nurture of mother earth cut off once and for all.'

[STORR'S VERGIL—*See Page* 16.]

36. ἵνα φάγῃ] In modern Greek, which properly speaking has no infinitive, the sense of the infinitive is expressed by νά (ἵνα) with subjunctive (as in this passage), *e.g.* ἐπιθυμῶ νὰ γράφῃ, 'I wish him to write;' see Corfe's *Modern Greek Grammar*, p. 78. This extension of the force of ἵνα to oblique petition, and even to consecutive clauses, may be partly due to the influence of the Latin *ut;* cf. ch. xvi. 27, ἐρωτῶ οὖν, πάτερ, ἵνα πέμψῃς: see note on ch. iv. 3.

The following incident is recorded by St. Luke alone. Simon the Pharisee is not to be identified with Simon the leper, Matt. xxvi., Mark xiv. 3.

ἀνεκλίθη] The Jews had adopted the Roman, or rather Greek, fashion of reclining at meals—a sign of advancing luxury and of Hellenism, in which however even the Pharisee acquiesces.

37. γυνή] There is no proof that this woman was Mary Magdalene. But mediæval art has identified the two, and great pictures have almost disarmed argument in this as in other incidents of the gospel narrative.

38. ἀλάβαστρον] The neuter sing. is Hellenistic. The classical form is ἀλάβαστρος with a heteroclite plural ἀλάβαστρα, hence probably the late sing. ἀλάβαστρον. The grammarian stage of a language loves uniformity, Herod. iii. 20; Theocr. xv. 114:

> Συρίῳ δὲ μύρῳ χρύσει' ἀλάβαστρα.

στᾶσα παρὰ τοὺς πόδας αὐτοῦ] This would be possible from the arrangement of the triclinium.

39. ἐγίνωσκεν ἄν] 'Would (all the while) have been recognising.'

40. χρεωφειλέται] A late word; the form varies between χρεωφειλέται and χρεοφειλέται.

41. δηνάρια] The denarius was a silver coin originally containing ten ases (deni), afterwards, when the weight of the as was reduced, sixteen ases. Its equivalent modern value is reckoned at 7½d. But such calculations are misleading; it is more to the point to regard the denarius as an average day's pay for a labourer.

42. μὴ ἐχόντων] Because *he saw that* they had not.

ἐχαρίσατο] Cf. *v.* 21.

[Carr's Notes on the Greek Testament—*See Page* 23.]

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